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# BASIC OF CIRCUIT THEORY

Transitional processes in electric circuits. Two – ports. Filters.
Manual

MINISTRI OF EDUCATION AND SCIENCE OF UKRAINE
National Aviation University

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## Transitional processes in electric circuits. Two – ports. Filters.

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Approved be the Board of the National Aviation University
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Посібник містить основні методи аналізу перехідних процесів у лінійних та нелінійних колах. Розглянуті чотириполюсники та фільтри на їх основі. Наведені приклади розрахунків та задачі для самостійного вивчення.

Для студентів напряму підготовки «Електронні пристрої та системи».

Pyanykh B. Ye.
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ISBH

It's expounding analysis of transient processes in linear and nonlinear electric circuits. It's considered classical and operational methods of calculation and method Duhamel integral. Its analyses transient processes particularity in nonlinear circuit and given examples of calculation by methods of graphic approximation, graphic integration, phase plane, successive approximations, attended intervals, space of state and another. It's separately selected transient processes in long lines. Considerable attention are considered analyses of two-poles and frequency filters on the their bases. Give an examples.

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## INTRODUCTION

The modern electronic use diverse circuits and devices, work of which is based on commutations – instantaneous changes parameters of elements, configuration of the circuit or input actions their characteristic. Transient processes are arise in the circuit, as result signals of definite form are formed and parameters.

Particularity of this item is analyses of transient processes in nonlinear electric circuits, on its contemporary electronic and electrical engineering is based. Including nonlinear circuits into context of the manual allow to widen boundary educational material into region, where known methods are non really or demand principle new approaches to the analyses and calculations. Modern of today demands. Theory of two ports is effectively used for analyze classical circuit of reactive filters, which are widely spread in radiotechnic and radioelecthonic devices. Methods of two - port theory are utilize in the circuit with distributed parameters, which have the great meaning by development nanotechnology, microelectronics, circuit engineering and making devices on their principles.

Study of discipline «Theory of electric and electronic circuit» demands solid preparation in sphere mathematics, physics, methods of analyze processes in complicated systems.

Material of a given manual is stated intelligible with sufficiently accuracy, which allow students successfully acquire material of manual.

# 1. KLASSICAL METHOD OF TRANSIENT PROCESSES ANALYSIS IN THE LINEAR CIRCUIT

## 1.1. General information abaut transient processes

In electric circuits distinguish modes:

1) is established, if currents and voltages do not change or change periodically;

2) transitional - in the transition from one steady mode to another.

The transition process occurs as a result of commutations - jump-like changes in parameters, configuration, circle structure, or input influences. It is believed that switching occurs instantaneously, and the transition process continues indefinitely.

The emergence of transients in circles with reactive elements (inductance $L$, capacitance $C$) is due to the impossibility of instantaneous change in the energy accumulated in them ($w_L = \frac{Li^2}{2}$, $w_C = \frac{Cu^2}{2}$), which otherwise would correspond to infinite power ($p == \frac{dw}{dt} \to \infty$).

In the absence of a group of reactive elements, transient processes do not occur (occur instantaneously).

## 1.2. Laws of switching and initial conditions

First switching law: current in inductance can not instantaneously change:

$$i_L(0-) = i_L(0) = i_L(0+). \tag{1.1}$$

Second switching law: the voltage on the capacitance can not be changed instantaneously:

$$u_C(0-) = u_C(0) = u_C(0+). \tag{1.2}$$

In formulas (1.1) and (1.2): $i_L(0-)$, $u_C(0-)$, $i_L(0)$, $u_C(0)$, $i_L(0+)$, $u_C(0+)$ – currents in the inductance and voltages on the capacitance immediately before switching, at the moment of switching and immediately after switching respectively.

The value of currents and voltages at the time $t = (0+)$, which occurs immediately after switching, are called initial conditions. Initial conditions are dependent and independent, zero and nonzero.

Dependent initial conditions are the currents and voltages that change at the moment of switching, for example, the voltage in the inductance $u_L(0+)$, and current in the capacitance $i_C(0+)$.

Independent initial conditions – currents and voltages which do not change at the moment of switching, for example, current in inductance $i_L(0+)$ and the voltage on the capacitance $u_C(0+)$.

If currents $i(0+) = 0$ and voltages $u(0+) = 0$, then they are called zero initial conditions, and if they aren't zero $i(0+) \neq 0$, $u(0+) \neq 0$ – nonzero.

## 1.3. The general approach to the analysis of transients by the classical method

By analysis of transients in electrical circuit by classical method in common case consists a system of linear differential acuation of the *n*-th order

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = \tag{1.3}$$
$$= b_m \frac{d^m f}{dt^m} + b_{m-1} \frac{d^{m-1} f}{dt^{m-1}} + \cdots + b_1 \frac{df}{dt} + b_0 f,$$

where $a_0$, $a_1$,…$a_n$, $b_0$, $b_1$,…$b_m$ – constant coefficients, which are determined only by the scheme of the circuits and its parameters; $x$, $f$ – the output (current or voltage) and input (voltage source or current) quantities respectively.

The order of the highest derivative in the equation (1.3) determines the order of the circuit. So, for example, if $n = 1$, then this is the circuit of the first order, etc.

Approximately the order of the circuit can be determined by the total number of reactive elements of the circuit scimes.

The solution of the system (1.3) is written as a sum of free $x_{fre}$ and forced $x_{for}$ components

$$x = x_{fre} + x_{for}. \tag{1.4}$$

The component $x_{fre}$ corresponds to the processes occurring in the circuit due to the difference in the energies of the reactive elements in one and the other established operating modes. In real circuits, in the presence of losses, free processes are damping, i.e.

$$\lim_{t\to\infty} x_{fre} = 0. \quad (1.5)$$

The free component is defined as the general solution of a homogeneous (without the right-hand side) differential equation

$$a_n \frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0 \quad (1.6)$$

The solution of equation (1.6) has the form

$$x_{fre} = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t} = \sum_{k=1}^{n} A_k e^{p_k t}, \quad (1.7)$$

where $A_1$, $A_2$,…, $A_k$,…, $A_n$ – the integration constants, $p_1$, $p_2$,…, $p_k$,…, $p_n$ – the roots of the characteristic equation:

$$a_n p^n + a_{n-1}p^{n-1} + \cdots + a_k p^k + \cdots + a_1 p + a_0 = 0. \quad (1.8)$$

The roots $p_k$ of the characteristic equation (1.8) for passive electric circuits are always valid, negative or complex with a negative valid part. The imaginary roots correspond to lossless circuits, in which transient processes do not attenuate.

The component $x_{for}$ in equation (1.4) corresponds to the steady-state conditions in the circuit after switching under condition (1.5). It is defined as a partial solution of the inhomogeneous (with right-hand side) differential equation (1.3).

### 1.4. General procedure for calculating transitional processes by the classical method

The calculation of transient processes is carried out in this order.

1. Make a differential equation for the post-commutation circuit with respect to the quantities, that are subject to the laws of commutation (current in inductance or voltage on capacitance) analogously to equation (1.3).

2. Find the free component $x_{fre}$ of the transition process. To do this, compile and solve the characteristic equation in the same way as the equation (1.8). Substitute the roots of the characteristic equation into the general solution (1.7) of the homogeneous characteristic equation (1.6).

3. Find the forced component $x_{for}$ by calculating the post-commutation circle in the steady state conditions.

4. Find the desired quantities as the sum of free and forced components.

5. Find independent initial conditions (current in inductance, voltage on the capacitance) by calculating to a commutative circuit in the steady state conditions.

6. Find a constant of integration from the initial conditions.

7. Record the final solution of the output differential equation.

## 1.5 Transition processes in first-order circuits

***Switching on rC - circuits for constant voltage***. Let's analyze the process of switching $rC$ - circuits on a constant voltage (Fig.1.1) according to the given calculation procedure.

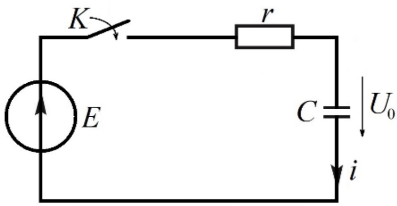



Fig. 1.1

By Kirchoff's law for the voltages for the circuit formed after the key $K$ is closed, we have $ri + u_C - E = 0$.

Because

$$i = C\frac{du_C}{dt},$$

then

$$rC\frac{du_C}{dt} + u_C = E \tag{1.9}$$

From equation (1.9), taking $\frac{du_C}{dt} = p$, $E = 0$, we have the characteristic equation:

$$rCp + 1 = 0 \tag{1.10}$$

The second term on the left-hand side of equation (1.9) has a zero-order derivative, so it is replaced by:

$$u_C = p^0 = 1. \tag{1.11}$$

The root of the characteristic equation

$$p = -\frac{1}{rC} = -\frac{1}{\tau}, \tag{1.12}$$

where $\tau = rC$ is the time constant.

Then the free component of the voltage on the capacitance

$$u_{fre} = Ae^{pt} = Ae^{-\frac{1}{\tau}}, \tag{1.13}$$

where $A$ is integration constant.

In steady state regime for the aftercommutation circuit

$$i = 0, u_C = E,$$

that is the forced component

$$u_{Cfor} = E.$$

Now the voltage on the capacitance for any moment of time, that is, the general solution of equation (1.9), can be found as the sum of free and forced components:

$$u_C = u_{Cfre} + u_{Cfor} = Ae^{-\frac{1}{\tau}} + E. \tag{1.14}$$

Independent initial conditions for this circuit are the voltage at the capacitance at the moment $t = (0+)$, ie immediately after switching . In according second commutation  low (1.2) we have

$$u_C(0+) = u_C(0-),$$

that is, you can find the voltage $u_C(0-) = U_0$ on the capacitance $C$, to which it will be charged in the pre-commutation circuit. so

$$u_C(0+) = u_C(0-) = U_0. \tag{1.15}$$

We substitute in equation (1.14) $t = 0$ and $u_C(0) = U_0$ from expression (1.15):

$$U_0 = A + E,$$

where

$$A = U_0 - E. \tag{1.16}$$

The final solution to the equations (1.15), (1.16):

$$u_C = (U_0 - E)e^{-\frac{1}{\tau}} + E, \tag{1.17}$$

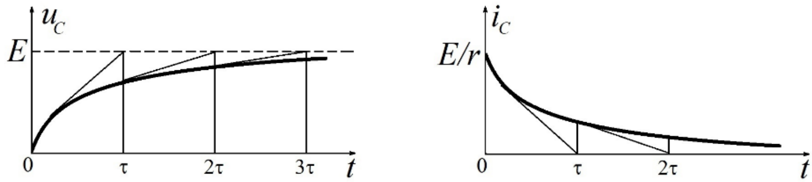
$$i_C = C\frac{du_C}{dt} = \frac{E - U_0}{r}e^{-\frac{1}{\tau}}. \tag{1.18}$$

The following  regimes are possible in Figure 1.1.

1. *Regime at* $U_0 = 0$. Then

$$u_C = E\left(1 - e^{-\frac{1}{\tau}}\right); i_C = \frac{E}{r}e^{-\frac{1}{\tau}}.$$

This is regime of activation of the unchaged capacitor at constant voltage $E$. The voltage exponentially increases from zero to $E$ (fig 1.2, a), the current at $t$ = 0 the jump increases to E/r and then exponentially decreases to zero (Fig. 1.2 b).




a b

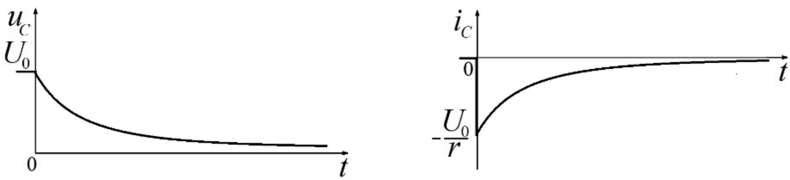
Fig.1.2

2. *Regime at* $E = 0$. Then

$$u_C = U_0 e^{-\frac{1}{\tau}};\ \ i_C = -\frac{U_0}{r}e^{-\frac{1}{\tau}}.$$

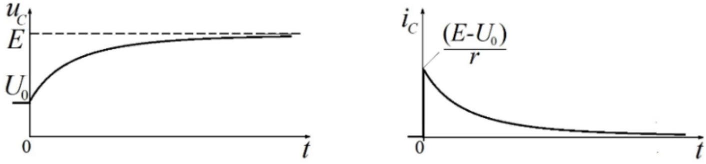
This is a free regime in the *rC*- circuit in which the capacitor, charged to the voltage $U_0$, is completely discharged with the development of the transition process (Fig. 1.3, a). The current $i_C$ changes the direction, the jump increases to the value $-\frac{U_0}{r}$ and exponentially decreases to zero (fig. 1.3, b).

3. *Regime at* $E > U_0$. Then the capacitor is charged from the voltage $U_0$ to $E$ according to equation (1.17) (Fig. 1.4 a), the jump increases from zero to and decreases to zero according to equation (1.18) (Fig. 1.4 b).




a b

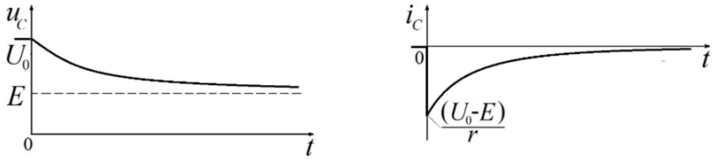
Fig. 1.3




a b

Fig. 1.4

4. *Regime at* $E < U_0$. Then the capacitor discharges from $U_0$ to $E$ (Fig. 1.5, a), and the current changes the sign, the jump increases to the value $\frac{U_0 - E}{r}$ and exponentially reduces to zero (Fig.1.5, b). Here are also the equations (1.17) and (1.18) for the voltage on the capacitance and current in it.




a b

Fig. 1.5

***Switching rC-circuits for harmonic voltage***. The ratio for transients according to the general calculation procedure can be obtained the same as in the case of switching $rC$-circuit to constant voltage at $E = e(t) = E_m \sin(\omega t + \varphi)$, where $\omega$, $\varphi$ – the angular frequency and the initial phase of harmonic electromotive force $e(t)$ (EMF). Therefore, $ri + u_C - e(t) = 0$. Hence $rC\frac{du_C}{dt} + u_C = E_m \sin(\omega t + \varphi)$. Here, as in formula (1.10) $rCp + 1 = 0$. Therefore, relation (1.10) - (1.13) are correct.

In the steady state conditions for the post-commutation circuit, the forced component is determined by the method of complex amplitudes.

Maximum circuit current

$$\dot{I}_{cm.for} = \frac{\dot{E}_m}{Z} = \frac{E_m e^{j\varphi}}{r + \frac{1}{j\omega C}} = \frac{j\omega C E_m e^{j\varphi}}{1 + j\omega rC} =$$

$$= \frac{\omega C E_m}{\sqrt{1 + (\omega rC)^2}} e^{j\left(\frac{\pi}{2} + \varphi - arctg\ \omega rC\right)}.$$

Voltage on capacitance

$$\dot{U}_{cm.for} = \dot{I}_{cm.for}\frac{1}{j\omega C} = \frac{E_m e^{j\varphi}}{r + j\omega C} =$$
$$= \frac{E_m}{\sqrt{1 + (\omega rC)^2}} e^{j(\varphi - arctg\ \omega rC)}.$$

Now

$$u_C = u_{Cfre} + u_{Cfor} = Ae^{-\frac{1}{\tau}} + \frac{E_m}{\sqrt{1 + (\omega rC)^2}} \times$$
$$\times\ sin(\omega t + \varphi - arctg\ \omega rC)\,. \qquad (1.19)$$

For initial conditions in a circuit the expression(1.15) is correct.

With equations (1.15), (1.19) we find at $t = 0$

$$U_0 = A + \frac{E_m}{\sqrt{1 + (\omega\tau)^2}} \sin(\varphi - \operatorname{arctg} \omega\tau)\,,$$

where

$$A = U_0 - \frac{E_m}{\sqrt{1 + (\omega\tau)^2}} \sin(\varphi - \operatorname{arctg} \omega\tau)\,. \qquad (1.20)$$

The final solution of equations (1.19) and (1.20) is:

$$u_C = U_0 e^{-\frac{1}{\tau}} + \frac{E_m}{\sqrt{1 + (\omega\tau)^2}} \times \qquad (1.21)$$
$$\times \left[\sin(\omega t + \varphi - \operatorname{arctg} \omega rC) - \sin(\varphi - \operatorname{arctg} \omega\tau) e^{-\frac{1}{\tau}}\right].$$

Here are the following modes:

*Regime at* $U_0 = 0$. This is the regime for switching the unchanged capacitor to harmonic voltage. From the equations (1.20) and (1.21)

$$A = -\frac{E_m}{\sqrt{1 + (\omega\tau)^2}} \sin(\varphi - \operatorname{arctg} \omega\tau)\,;$$

$$u_C = -\frac{E_m}{\sqrt{1 + (\omega\tau)^2}} \times$$
$$\times \left[\sin(\omega t + \varphi - \operatorname{arctg} \omega rC) - \sin(\varphi - \operatorname{arctg} \omega\tau) e^{-\frac{1}{\tau}}\right].$$

Obviously, the transition process is absent, since $A = 0$. The transition process takes place with $A \neq 0$. If $\varphi = \frac{\pi}{2} + \operatorname{arctg} \omega\tau$ then, the integration constant became the maximum:

$$A = -\frac{E_m}{\sqrt{1 + (\omega\tau)^2}}$$

and then

$$u_C = -\frac{E_m}{\sqrt{1 + (\omega\tau)^2}}\left(\cos \omega t - e^{-\frac{1}{\tau}}\right).$$

The graph of the transition process is shown in Fig.1.6, from which it is evident that the voltage on the capacitor can significantly exceed the established voltage value. The maximum voltage is due to the half-life of the harmonic voltage from the moment of switching.

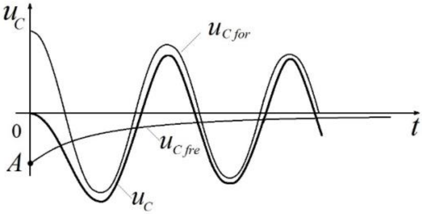



Fig.1.6

*Regime at* $U_0 \neq 0$. Then at $\varphi = \operatorname{arctg} \omega\tau$ we have $A = U_0$. There is a transient process appies according to the equation (1.19):

$$u_C = U_0 e^{-\frac{1}{\tau}} + \frac{E_m}{\sqrt{1 + (\omega\tau)^2}} \sin \omega t.$$

Graph of it is images in Fig. 1.7.

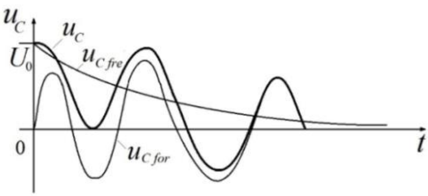



Fig.1.7

***The time constant and duration of the transition process***. The values $\tau = rC$ or $\tau = L/r$ are called time constants. Measured in seconds (s).

If

$$\left(e^{-\frac{1}{\tau}}\right)' = -\frac{1}{\tau}\left(e^{-\frac{1}{\tau}}\right),$$

and with $t = 0$

$$\left(e^{-\frac{1}{\tau}}\right)' = -\frac{1}{\tau},$$

then the component τ is equal to the length under the tangent, conducted to the exponential curve (see Fig. 1.2).

At $t = \tau$

$$e^{-\frac{1}{\tau}} = e^{-1} = 0{,}367,$$

that is τ becomes equal to the time for which the free component is changed in $e$ times, that is to $0{,}367$ from its value at the beginning of the interval.

The transition process is considered to be practically completed in a while $t_{tr} = (3 - 5)\,\tau$. During this time, the exponent reaches (95 - 99)% its value in the steady state conditions.

***Qualitative analysis of transient processes.*** For circuit of the first order, it is convenient to carry out a qualitative analysis of transient processes without compiling and solving differential equations, determining the currents and voltages on the elements:

- before the switching circuit with $t = (0-)$,
- aftercommutational circuit with $t = (0+)$,
- circuits in steady state condition $t \ \rightarrow \infty$.

**Example 1.1.**

Calculate transient process for the circuit fig.E.1.8 by qualitative method.

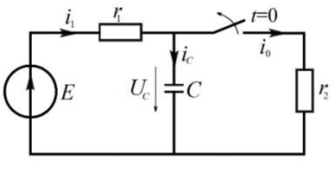



Fig.E.1.8

In according calculation order

1) $t = (0-); i_1 = i_2 = \frac{E}{r_1+r_2}; i_C = 0; u_C(0) = \frac{Er_2}{r_1+r_2};$
2) $t = (0+); i_1 = i_2 = \frac{E - u_C(0)}{r_1} = \frac{E}{r_1+r_2}; i_2 = 0;$

$u_C(0+) = u_C(0-) = \frac{Er_2}{r_1+r_2};$

3) $t \to \infty$; $i_1 = i_2 = i_C = 0$; $u_C = E$.

Graphic of transient process for the currents $i_1$, $i_2$, $i_C$ are shown in fig.E.1.9,a,b,c and for voltage $u_C$ - in fig.E.1.9,d.

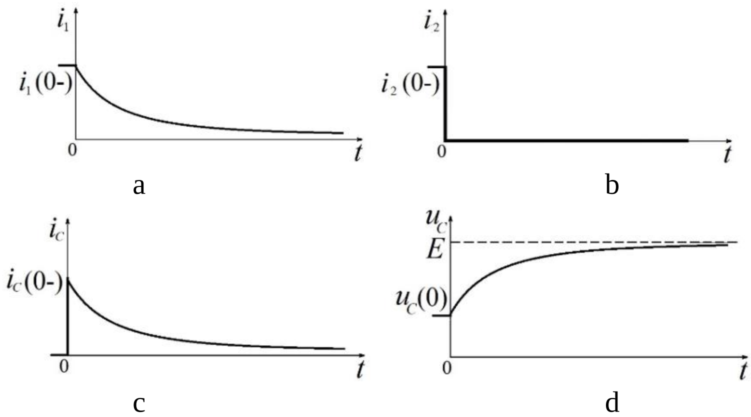



Fig.E.1.9

**Problem 1.1.**

Calculate and analyze transient processes in a given linear circuit of the second order which source of constant EMF(fig. P.1.10) by classical method. It's given:

$E = 100$ V, $L = 1$ mH, $C = 10$ mcF, $r_1 = 10$ Ohms,

$r_2 = 10$ Ohms, $r_3 = 4$ Ohms.

Find $i_1$.

*Solution.*

1. After commutation circuit is shows in fig.P.1.11. This circuit has two nodes. The lows node is taken as basis, is counted from voltage $u_C$. Let's compile equation in according current Kirchhoff's low for the node 1.

$$i + i_1 + i_2 + i_3 = 0 \qquad \text{(P.1.22)}$$

Here:

$$i = \frac{u_C - E}{r_1 + r_2}; \quad i_1 = \frac{u_C}{r_3};$$

$$i_2 = \frac{1}{L}\int u_C\, dt, \left(\text{as } u_C = u_L = L\frac{di}{dt}\right); i_3 = C\frac{du_C}{dt}. \qquad \text{(P.1.23)}$$

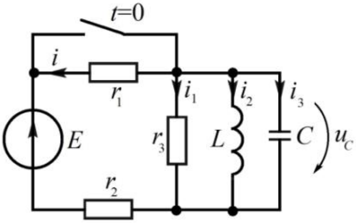


Fig.P.1.10

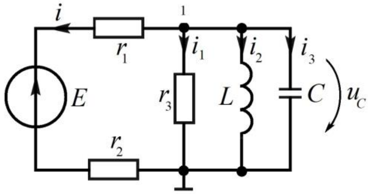


Fig. P.1.11

Let's substitute (P.1.23) into (P.1.22). We get

$$\frac{u_C - E}{r_1 + r_2} + \frac{u_C}{r_3} + \frac{1}{L}\int u_C\,dt + C\frac{du_C}{dt} = 0. \qquad \text{(P.1.24)}$$

Differential relatively time $t$ gives

$$\frac{d^2u_C}{dt^2} + \frac{r_1 + r_2 + r_3}{(r_1 + r_2)r_3C}\frac{du_C}{dt} + \frac{1}{LC}u_C = 0. \qquad \text{(P.1.25)}$$

Let's designate

$$\frac{r_1 + r_2 + r_3}{(r_1 + r_2)r_3C} = \frac{1}{\tau} = 2\delta;\ \ \tau = \frac{(r_1 + r_2)r_3C}{r_1 + r_2 + r_3};\ \ \frac{1}{\sqrt{LC}} = \omega_0.$$

We receive from (P.1.25)

$$\frac{d^2u_C}{dt^2} + 2\delta\frac{du_C}{dt} + \omega_0^2u_C = 0. \qquad \text{(P.1.26)}$$

Equation (P.1.26) is differential equation for after commutation circuit in fig. P.1.11 relatively value $u_C$, which obey commutation lows.

1. Introduce substitution: $\frac{d^2u_C}{dt^2} = p^2$; $\frac{du_C}{dt} = p^1$; $u_C = p^0 = 1$, we receive characteristically equation

$$p^2 + 2\delta p + \omega_0^2 = 0. \qquad \text{(P.1.27)}$$

Ruts of equation (P.1.27)

$$p_1 = -\delta + \sqrt{\delta^2 - \omega_0^2};\, p_1 = -\delta - \sqrt{\delta^2 - \omega_0^2}. \qquad \text{(P.1.28)}$$

Free component voltage across capacitance is

$$u_{Cfre} = A_1e^{p_1t} + A_2e^{p_2t}. \qquad \text{(P.1.29)}$$

In force regime inductance $L$ in (fig.P.11.) shunts capacitance $C$. Therefore

$$u_{Cfor} = 0 \qquad \text{(P.1.30)}$$

2. Before commutation circuit is shown in fig.P.1.12. In before commutation circuit inductance $L$ shunts capacitance $C$. Therefore

$$u_C(0) = 0. \qquad \text{(P.1.31)}$$

That is initial value voltage across capacitance.

3. Full voltage across capacitance in according (P.1.29), (P.1.30)

$$u_C = u_{Cfre} = A_1 e^{p_1 t} + A_2 e^{p_2 t}. \qquad \text{(P.1.32)}$$

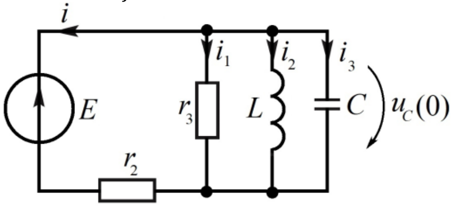



Fig. P.1.12

4. For finding integration constants $A_1$ and $A_2$ it is necessary to supplement (P.1.22) else one equation. Find derivative of $u_C$ from (P.1.32)

$$\frac{du_C}{dt} = A_1 p_1 e^{p_1 t} + A_2 p_2 e^{p_2 t}. \qquad \text{(P.1.33)}$$

Value

$$C\frac{du_C}{dt} = i_C = i_3. \qquad \text{(P.1.34)}$$

If $u_C(0) = 0$, then at $t = 0$ we get from fig P.1.11

$$i_1(0) = 0. \qquad \text{(P.1.35)}$$

It is obvious, according fig. P.1.12

$$i2(0-) = i2(0) = i2(0+) = \frac{E}{r_2}. \qquad \text{(P.1.36)}$$

For the loop $E$, $r_1$, $C$, $r_2$ in fig. P.1.11 we get.

$$-E - i(r1 + r2) + uC = 0.$$

Hence at $t = 0$

$$i(0) = -\frac{E}{r_1 + r_2}. \qquad \text{(P.1.37)}$$

Then, according (P.1.22) at $t = 0$ take into account (P.1.35), (P.1.36), (P.1.37) we get

$$i(0) + i_1(0) + i_2(0) + i_3(0) = -\frac{E}{r_1 + r_2} + \frac{E}{r_2} + i3(0) = 0. \qquad \text{(P.1.38)}$$

and

$$i_3(0) = \frac{E}{r_1 + r_2} - \frac{E}{r_2} = -\frac{E r_1}{(r_1 + r_2) r_2}.$$

Let’s multiply (P.1.22) by $C$. Then, using (P.1.34), (P.1.38) at $t = 0$, we get second equation for definition $A_1$, $A_2$

$$i_3(0) = C(A_1p_1 + A_2p_2). \qquad \text{(P.1.39)}$$

Now, using (P.1.131) - (P.1.34), (P.1.39), (P.1.37) we get equation system

$$\begin{cases} A_1 + A_2 = 0; \\ C(A_1p_1 + A_2p_2) = -\dfrac{Er_1}{(r_1 + r_2)r_2}. \end{cases}$$

Whence

$$A_1 = \frac{Er_1}{C(r_1 + r_2)(p_2 - p_1)r_2};$$

$$A_2 = -\frac{Er_1}{C(r_1 + r_2)(p_2 - p_1)r_2}.$$

Now

$$u_C = \frac{Er_1}{C(r_1 + r_2)(p_2 - p_1)r_2}e^{p_1t} - \frac{Er_1}{C(r_1 + r_2)(p_2 - p_1)r_2}e^{p_2t} =$$

$$= \frac{Er_1}{C(r_1 + r_2)(p_2 - p_1)r_2}(e^{p_1t} - e^{p_2t}).$$

That is so

$$i_1 = \frac{u_C}{r_3}$$

then

$$i_1 = \frac{Er_1}{Cr_3(r_1 + r_2)(p_2 - p_1)r_2}(e^{p_1t} - e^{p_2t}). \qquad \text{(P.1.40)}$$

## Methodic instruction

It’s necessary account of two kinds transient regimes in electrical circuits: steady – state and transient condition by study material “Classical method of transient processes analyses”. Currents and voltages by steady – state condition aren’t change or periodic change. Elements of transient conditions are absent in harmonic current circuits. But steady – state condition may be include periodic transient processes in circuit of non harmonic current (nonlinear or parametrical circuit). Notion “transient condition” envelope any independent transient regimes in such circuits. The given section study only one kind of transient process, which take place in all elements of linear electrical circuits.

It’s necessary to considered commutation lows, expound the common order of transient processes calculation in according of a given algorithm and examples. Study of this material demands repetition the methods of mathematical analyses for solution linear differential equation.

Material about analysis and calculation transient processes in nonlinear circuit is given in this section. Particularity of nonlinear circuits and methods of numerical calculations are considerate in this section.

Literature: [ 1,2,6,8 – 10, 13 – 15]

## Questions for self checking

1. Let’s name reasons of transient processes beginning.
2. Formulate commutation lows.
3. What are initial conditions?
4. Let’s name the base points of common order classical method of transient processes analyses.
5. What is order of quality transient processes analyses?
6. What is integrate method approximation for the transient processes in nonlinear circuit?
7. What is graphic integration method for the transient processes in nonlinear circuit calculation?
8. What is method of phase plane for the transient processes in nonlinear circuit calculation?
9. What is method of successive approximation for the transient processes in nonlinear circuit calculation?
10. What is mating intervals method for the transient processes in nonlinear circuit calculation?
11. What is fined increment method for the transient processes in nonlinear circuit calculation?
12. What is method of state space for the transient processes in nonlinear circuit calculation?
13. What are methods of averaging for the transient processes in nonlinear circuit calculation?

# 2. OPERATIONAL METHOD OF TRANSIENT PROCESSES ANALYSIS

## 2.1. Common information about operational method

Transient processes analysis foresees solution differential equations of electrical balance in the circuit. For such solution the operational method is widely used, which is based on Laplace transforms. By that function of real variable $t$ replaces by function complex variable $p = \sigma + j\omega$. As result differential equations is substitute by algebraic equations and after solution reverse transmission is gives real variable $t$.

For the first time this was shown by M. Ye. Vashcenko-Zakharchenko of Russia in his monograph, *Simvolichedkoe ischislenie i prilozenie ego k integrirovaniu lineynykh differentsialnykh uravneniy* (Simbolic Calculus and Its Application to the Integration of Linear Differential Equations) (Kiev, 1862). Independently of him, O. Heaviside of England at the end of the century proposed the use of operational calculus to the analyses of electromagnetic transients. However, Heaviside did not set forth any mathematical principles underlying the method. Further progress in the use of the operational method has been due to many scientists, among them. V. S. Ignatovsky, D. R. Carson, B. van-der Pol, A. M. Efros, A. M. Danilovsky, K. A. Krug and A. I. Lurje, to name but few. Vashcenko- Zakharchenko also the showed that the operational method could de applied not only to ordinary linear differential equations with constant coefficients and their systems, but also to linear equations with constant coefficients and variable coefficients and to partial differential equations with constant coefficients or, in term of electrical engineering, to the transient analysis of distributed-parameter circuits.

The operation method consists in that the given univalued bounded function of a real variable, say, time (that is, $f(t)$), called the *original function*, satisfying Dirichlet's conditions over any finite time interval and equal to zero at $t < 0$ is transformed to another function, $F(p)$, of a complex frequency $p = \sigma + j\omega$.

The new function is called the Laplace transform of the original time function.

As will be recalled, Dirichlet's conditions require that over any finite interval the function $f(t)$ should be either continuous or have a finite number of maxima and minima over that interval.

Transition from $t$ to $p$ is cold direct Laplace transforms ($L$ – laplacian).

$$L[f(t)] = F(p) = \int_0^{\infty} f(t)e^{-pt}\,dt. \qquad (2.1)$$

Reverse transforms from $p$ to $t$ is cold reverse Laplace transforms

$$L^{-1}[f(t)] = f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(p)e^{-pt}dt\,. \qquad (2.2)$$

Integral (2.2) is cold Bromwich integral. Function $f(t)$ is cold original, $F(p)$ is image.

Let's consider the some properties of Laplace transforms.

1. Image of constant $A$. In according direct Lap lace transform

$$L(A) = \int_0^{\infty} Ae^{-pt}\,dt = A\left(-\frac{1}{p}\right)\int_0^{\infty} d(e^{-pt}) = \frac{A}{p}.$$

That is way

$$L(A) = \frac{A}{p} \text{ and } L(1) = \frac{1}{p}. \qquad (2.3)$$

2. Multiplication of function $f(t)$ by constant $A$. In according direct Laplace transform we get

$$L[Af(t)] = \int_0^{\infty} A\,f(t)e^{-pt}\,dt = A\int_0^{\infty} f(t)e^{-pt}\,dt = AF(p).$$

That is way

$$L[Af(t)] = AF(p)$$

3. Linearity

$$L[f1(t) \mp f2(t)] = \int_0^{\infty} f_1(t)e^{-pt}\,dt \pm \int_0^{\infty} f_2(t)e^{-pt}\,dt =$$
$$= F_1(p) \pm F_2(p),$$

than

$$L[f1(t) \mp f2(t)] = F_1(p) \pm F_2(p).$$

4. Image of derivative

$$L\left\{\frac{d[f(t)]}{dt}\right\} = \int\limits_0^\infty \frac{\{d[f(t)]}{dt}\, e^{-pt}dt = \int\limits_0^\infty e^{-pt}\, d[f(t)]. \quad (2.4)$$

For integration by parts we have

$$\int u dv = uv - \int v du\,. \qquad (2.5)$$

Here $v = f(t)$, $du = (de^{-pt})$, than

$$L\left\{\frac{d[f(t)]}{dt}\right\} = \{e^{-pt}f(t)\}|_0^\infty - \int\limits_0^\infty f(t)d(e^{-pt}) =$$

$$= -f(0) - \int\limits_0^\infty f(t)e^{-pt}(-p)dt =$$

$$= -f(0) - p\int\limits_0^\infty f(t)e^{-pt}dt = -f(0) + pF(p).$$

That is way

$$L\left\{\frac{d[f(t)]}{dt}\right\} = pF(p) - f(0)$$

It's evidence

$$L\left\{\frac{d^2[f(t)]}{dt^2}\right\} = p^2F(p) - f'(0) - f(0).$$

That is way, image of derivative of the any order from time function is equal to product operator $p$ to the degree of derivative order by image of time function which precision about integration constant.

5. Image of integral $\int_0^t f(t)dt$. Accounting expression (2.1), we get

$$L\left[\int\limits_0^t f(t)dt\right] = \int\limits_0^\infty \left[\int\limits_0^t f(t)dt\right] e^{-pt}dt =$$

$$= -\frac{1}{p}\int\limits_0^\infty \left[\int\limits_0^t f(t)dt\right] d(e^{-pt}). \qquad (2.6)$$

Let's designation for integration by part

$$\int_0^t f(t)dt = u, \qquad d(e^{-pt}) = dv.$$

As $v = e^{-pt}$, $du = f(t)dt$.

Then from expressions (2.5), (2.6) we get

$$\int_0^t f(t)dt = -\frac{1}{p}\left[\int_0^t f(t)dt\, e^{-pt} \Big|_0^\infty\right] + \frac{1}{p}\int_0^\infty f(t)e^{-pt}dt =$$

$$= \frac{1}{p}\int_0^\infty f(t)e^{-pt}dt = \frac{F(p)}{p}.$$

That is way

$$L\left[\int_0^t f(t)dt\right] = \frac{F(p)}{p}.$$

It’s evidence

$$L\left[\iint_0^t f(t)dt\right] = \frac{F(p)}{p^2}.$$

That is way, image of integral of the any order from time function is equal to time function divisible by operator $p$ to degree of integral multiple.

6. Phase shift of original.

Function $f(t-\tau)$ is named time shift relatively function $f(t)$ at interval $\tau$ (Fig. 2.1). Property of phase shift is comfortable used by receiving image of function, which are given different expressions (**piece – continuous function**), for example, for analysis the complicated form signals passing through a linear electric circuit.

If $L[f(t)] = F(p)$, then we have

$$L[f(t-\tau)] = \int_0^\infty f(t-\tau)e^{-pt}dt =$$

$$= \int_0^\infty f(t-\tau)\, e^{-p(t-\tau)}e^{-p\tau}d(t-\tau) =$$

$$= e^{-p\tau}\int_0^\infty f(t-\tau)\; e^{-p(t-\tau)}\; d(t-\tau) = e^{-p\tau}F(p).$$

That is way

$$L[f(t-\tau)] = e^{-p\tau}F(p) \tag{2.7}$$

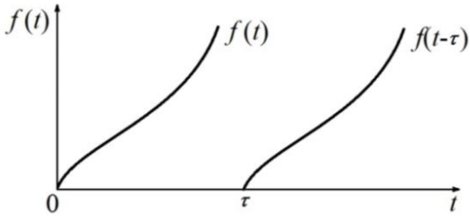



Fig. 2.1

7. Shift.

Function $f(t)e^{-\alpha t}$ is cold displacement relatively function $f(t)$ at angle α. In according (2.1) we get

$$L[f(t)e^{-\alpha t}] = \int_0^\infty f(t)e^{-\alpha t}e^{-pt}dt = \int_0^\infty f(t)e^{-(p+\alpha)t}dt = F(p+\alpha)$$

That is way rule of shift

$$L[f(t)e^{\pm\alpha t}] = F(p \mp \alpha). \tag{2.8}$$

That is way multiplication function $f(t)$ by $e^{\pm\alpha t}$ correspond substitution in image $p$ for $p \mp \alpha$.

8. Similarity (changing of scale independent variable).

Let's $a$ is any positive number. Then, if $L[f(t)] = F(p)$ we get

$$L[f(at)] = \int_0^\infty e^{-pt}f(at)dt.$$

For integration by parts we designation: $at = u$, $dt = \frac{du}{a}$. Then

$$L[f(at)] = \frac{1}{a}\int_0^\infty e^{-\frac{p}{a}u}f(u)du = \frac{1}{a}F\left(\frac{p}{a}\right).$$

Analogically

$$L(\sin t) = \int_0^\infty e^{-pt} \sin t dt = \frac{e^{-pt}(-p\sin t - \cos t)}{p^2 + 1}\Bigg|_\infty^0 = \frac{\omega}{p^2 + 1};$$

$$L(\cos t) = \int_0^\infty e^{-pt} \cos t dt = \frac{e^{-pt}(\sin t - p\cos t)}{p^2 + 1}\Bigg|_\infty^0 = \frac{p}{p^2 + 1};$$

or

$$L(\sin \omega t) = \frac{\omega}{p^2 + \omega^2}; \quad L(\cos \omega t) = \frac{p}{p^2 + \omega^2}.$$

Using properties of phase shift and similarity, we get images of functions $\sin(\omega t - \varphi)$ and $\cos(\omega t - \varphi)$. As

$$L[\sin(\omega t - \varphi)] = L[\sin \omega t \cos \varphi - \cos \omega t \sin \varphi] =$$
$$= \cos \varphi \frac{\omega}{p^2 + \omega^2} - \sin \varphi \frac{p}{p^2 + \omega^2};$$
$$L[\cos(\omega t - \varphi)] = L[\cos \omega t \cos \varphi + \sin \omega t \sin \varphi] =$$
$$= \cos \varphi \frac{p}{p^2 + \omega^2} + \sin \varphi \frac{\omega}{p^2 + \omega^2}.$$

9. Convolution of original (multiplication of images).

Convolution of continuous functions $f(t)$ and $\varphi(t)$ is named function $\psi(t)$, in according equality

$$\psi(t) = f(t) * \varphi(t) = \int_0^t f(\tau)\, \varphi(t - \tau) d\tau.$$

Convolution has the same properties as multiplication:

a) commutativity: $f * \varphi = \varphi * f$;

b) associativity: $(f * \varphi) * \psi = f * (\varphi * \psi)$;

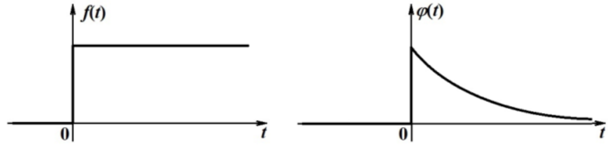
c) reflexivity: $(f + \varphi) * \psi = f * \psi + \varphi * \psi$.

That is way, convolution of function gives the same result independently on convolution order.

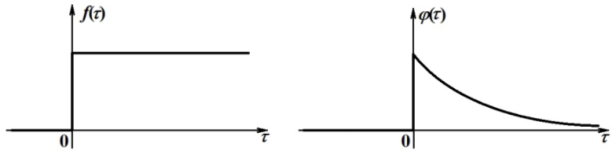
Convolution theorem: multiplication of two function images is accordance image of this function convolution.

Convolution of functions has graphic interpretation. Convolution functions of $f(t)$ with $\varphi(t)$ can by contribution as production of more than two cofactors. Than convolution property is used successive to in pairs rally around cofactors. It is shown convolution property grouping can by perform in any order.

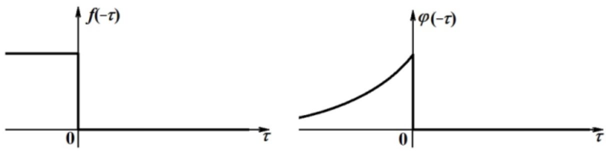
Convolution functions of $f(t)$ with $\varphi(t)$ can by represent graphically (fig. 2.2).



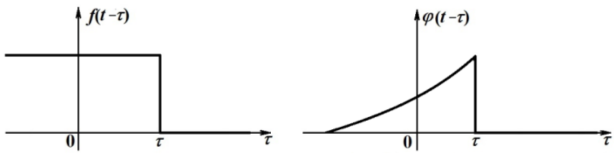

a



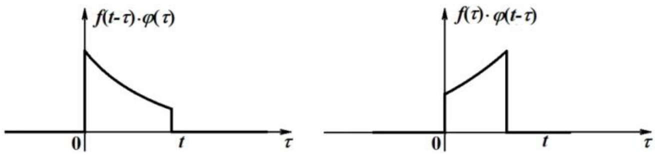

b



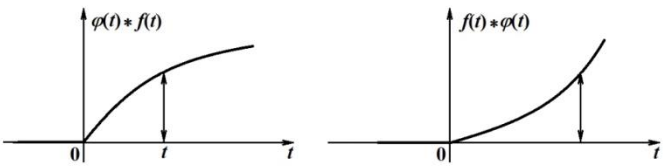

c




d




e




f

Fig. 2.2

Fig.2.2,a shows convolution of two functions $f(t)$ and $\varphi(t)$, in fig.2.2,b – functions $f(\tau)$ and $\varphi(\tau)$ after substitution of a variable $t$ to τ. Functions $f(-\tau), \varphi(-\tau)$ are show in Fig. 2.2,c. They are mirror reflection of functions $f(\tau)$ and $\varphi(\tau)$ relatively ordinate axes. Shift of functions $f(-\tau), \varphi(-\tau)$ to the right on value *t* is shown in Fig.2.2,d.

Product of functions $f(t-\tau) * \varphi(\tau)$ and $f(\tau) * \varphi(t-\tau)$ are shown in Fig.2.2,e. Integration without from 0 to τ gives areas, which is shaded in fig. 2.9, e. They areas are equal to convolution functions $f(t) * \varphi(t)$. Dependence convolution on time $t$ is shown in fig.2.2,f. From fig.2.9,e,f it is shown: Integrals of productions $f(t-\tau) * \varphi(\tau)$ and $f(\tau) * \varphi(t-\tau)$ are equal.

## 2.2. The decomposition formula

Using the decomposition formula, you can find the original $f(t)$ of the known image $F(p)$

$$F(p) = \frac{F_1(p)}{F_2(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + \dots + b_0}{a_n p^n + a_{n-1} p^{n-1} + \dots + a_0} \qquad (2.9)$$

where $F_1(p)$, $F_2(p)$ are polynomials of whole degrees $p$ ($m$ and $n$) at the same time $m < n$. The coefficients $a_k$, $b_k$ are valid and are determined only by the parameters of the circuit. Polynomials $F_1(p)$, $F_2(p)$ haven't common roots, that is, the fraction (2.9) is non-cancellable.

Expansion the fraction (2.9) into prime fractions. If $p_1, p_2, \dots, p_n$ – different roots of the polynomial F2(p), then

$$\frac{F_1(p)}{F_2(p)} = \frac{A_1}{p-p_1} + \frac{A_2}{p-p_2} + \dots + \frac{A_n}{p-p_n} = \sum_{k=1}^{n} \frac{A_k}{p-p_k}. \qquad (2.10)$$

Determine the coefficients of decomposition $A_1, A_2, \dots, A_n$. Multiply fraction (2.10) by $p - p_k$:

$$(p-p_k) \sum_{k=1}^{n} \frac{A_k}{p-p_k} = \frac{F_1(p)(p-p_k)}{F_2(p)}. \qquad (2.11)$$

We are heading $p \to p_k$. Because $p_k$ is the root of the $F_2(p)$, then

$$\lim_{p \to p_k} \frac{p-p_k}{F_2(p)} = \frac{0}{0}. \qquad (2.12)$$

We reveal the (2.12) by the Lopital rule:

$$\lim_{p \to p_k} \frac{(p - p_k)'}{F_2'(p)} = \frac{1}{F_2'(p_k)}. \tag{2.13}$$

Left part in expression (2.11)

$$\lim_{p \to p_k} (p - p_k) \sum_{k=1}^{n} \frac{A_k}{p - p_k} = A_k \,, \tag{2.14}$$

that is, according to expressions (2.11), (2.13) and (2.14) at $p \to p_k$

$$A_k = \frac{F_1(p_k)}{F_2'(p_k)}. \tag{2.15}$$

Then from equation (2.10) given (2.15) we have

$$\frac{F_1(p)}{F_2(p)} = \sum_{k=1}^{n} \frac{F_1(p_k)}{F_2'(p_k)(p - p_k)}. \tag{2.16}$$

Determine the original from the image (2.16). Since $F_1(p_k)$ and $F_2'(p_k)$ are steel quantities, it is necessary to find the original expression $\frac{1}{p-p_k}$. Given equations (2.3) and (2.8), we have

$$\frac{1}{p - p_k} = e^{p_k t}.$$

So, finally

$$F(p) = \frac{F_1(p)}{F_2(p)} \doteqdot \sum_{k=1}^{n} \frac{F_1(p_k)}{F_2'(p_k)} e^{p_k t} = f(t). \tag{2.17}$$

Expression (2.17) is a decomposition formula. If there are zero roots among the roots of a polynomial $F_2(p)$, that is

$$F_2(p) = pF_3(p),$$

where the polynomial $F_3(p)$, hasn't zero roots, then by the formula (2.17) we obtain:

$$F(p) = \frac{F_1(p)}{F_2(p)} \doteqdot \frac{F_1(0)}{\left\{\frac{d}{dp}[pF_3(p)]\right\}_{p=0}} + \sum_{k=1}^{n} \frac{F_1(p_k)}{\left\{\frac{d}{dp}[pF_3(p)]\right\}_{p=p_k}} e^{p_k t} =$$

$$= \frac{F_1(0)}{F_3(0)} + \sum_{k=1}^{n} \frac{F_1(p_k)}{p_k F_3'(p_k)} e^{p_k t} = f(t). \tag{2.18}$$

It takes into account that

$$\left\{\frac{d}{dp}[pF_3(p)]\right\}_{p=0} = [F_3(p) + pF_3'(p)]_{p=0} = F_3(0)$$

and

$$F_3(p_k) = 0.$$

The decomposition formula (2.17) can be written in general form

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{F_1(p)}{F_2(p)}e^{pt}dp \approx \sum_{k=1}^{n}\operatorname{Res}\left[\frac{F_1(p)}{F_2(p)}e^{pt}\right]_{p=p_k},$$

where the excess Res is defined as

$$\operatorname{Res}\left[\frac{F_1(p)}{F_2(p)}e^{pt}\right]_{p=p_k} = \frac{F_1(p_k)}{F_2'(p_k)}e^{p_k t}.$$

In the table. 2.1 the originals of the some features and their Laplace images are showed.

In the table. 2.1 shows the originals of some features and their Laplace images.

Table 2.1

| Original | Image |
|---|---|
| $1(t)$ | $\frac{1}{p}$ |
| $\delta(t)$ | $1$ |
| $e^{\pm at}$ | $\frac{1}{p+a}$ |
| $1-e^{-at}$ | $\frac{a}{p(p+a)}$ |
| $\frac{1}{b-a}\left(e^{-at}-e^{-bt}\right)$ | $\frac{1}{(p+a)(p+b)}$ |
| $\cos(\omega t+\psi)$ | $\frac{p\cos\psi-\omega\sin\psi}{p^2+\omega^2}$ |
| $\sin(\omega t+\psi)$ | $\frac{p\cos\psi+\omega\sin\psi}{p^2+\omega^2}$ |

As already noted, the decomposition formula applies only when $F_2'(p_k)$ it hasn't multiple roots. Indeed, for example, the image

$$F(p) = \frac{F_1(p)}{F_2(p)} = \frac{1}{(p+a)^2}, \tag{2.19}$$

the denominator of which $F_2(p)$, has two multiple roots $p_1 = p_2 = -a$, by the formula (2.17) gives for the original expression:

$$f(t) = \frac{1}{0}e^{-at} + \frac{1}{0}e^{-at} \to \infty. \qquad (2.20)$$

However, from the table 2.1 the original image (2.19) can be obtained from the ratio

$$\frac{1}{(p+a)(p+b)} = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)$$

by border crossing $b \to a$:

$$\lim_{b\to a}\frac{1}{(p+a)(p+b)} = \frac{1}{(p+a)^2} = \lim_{b\to a}\frac{1}{b-a}\left(e^{-at} - e^{-bt}\right) =$$

$$\lim_{b\to a}\frac{\frac{d}{da}\left(e^{-at} - e^{-bt}\right)}{\frac{d}{da}(b-a)} = te^{-at}, \qquad (2.21)$$

that is, the result is incorrect.

The original image for the multiple roots of the denominator can be found by the convolution property (Borel's theorem), the essence of which is. Let the image $F(p)$, be presented as a product

$$F(p) = F_1(p)F_2(p). \qquad (2.22)$$

By direct Laplace transform (2.1)

$$F_2(p) = \int_0^\infty f_2(\tau)\, e^{-p\tau} d\tau, \qquad (2.23)$$

so

$$F(p) = F_1(p)F_2(p) = F_1(p)\int_0^\infty f_2(\tau)\, e^{-p\tau} d\tau =$$

$$\int_0^\infty e^{-p\tau} F_1(p) f_2(\tau)\, d\tau. \qquad (2.24)$$

By delay property (2.7)

$$e^{-p\tau}F_1(p) = f_1(t-\tau).$$

Then the original image (2.7)

$$F(p) = f(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau. \quad (2.25)$$

The formula (2.25) is called the convolution formula. Obviously, the functions $F_1(p)$ та $F_2(p)$ are equal, therefore, performing similarly the transformation (2.23) for $F_1(p)$, we get

$$F(p) = f(t) = \int_0^t f_1(\tau) f_2(t-\tau) d\tau. \quad (2.26)$$

Let's determine the original image (2.19) with the convolution formula (2.25) with taking into account that

$$F(p) = F_1(p) F_2(p) = \frac{1}{p+a} \frac{1}{p+a}.$$

**Given the properties** of the shift operation (2.8))

$$\frac{1}{p+a} = e^{-at} = f_1(t) = f_2(t)$$

by the formula (2.25) we have

$$f(t) = \int_0^t e^{-a(t-\tau)} e^{-a\tau} d\tau = e^{-at} \int_0^t d\tau = e^{-at} \tau \Big|_0^t = t e^{-at},$$

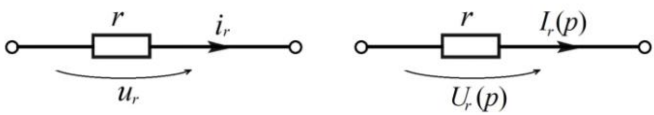
which coincides with the relation (2.21).

## 2.3. Operational substitution circuit of the basic circuit elements

***Active resistance***. For active electric resistance $r$ we can write

$$u_r = i_r r, \quad (2.27)$$

where $i_r$, $u_r$ – instantaneous values of electric current and voltage (Fig.2.2,a).




a b

Fig. 2.2

And in operational form it is written as

$$u_r \doteqdot U_r(p), i_r \doteqdot I_r(p). \qquad (2.28)$$

So, from expression (2.27) we get

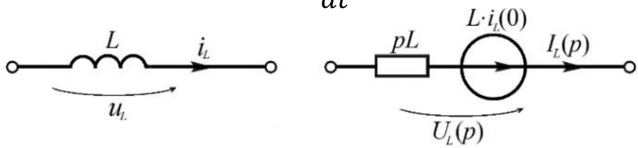
$$U_r(p) = I_r(p)r\ .$$

The second property of the Laplace transform is used here - the multiplication of the function $f(t)$ on a constant value.

Thus, the operator image of the resistance $r$ is the same resistance $r$ (Fig.2.2,b).

***Inductance***. For the inductance $L$ (Fig. 2.2,a) the ratio between the instantaneous values of current $i_L$ and the voltage $u_L$ can be written as

$$u_L = L\frac{di_L}{dt}.$$




a b

Fig. 2.3

In the operator form it can be written similar to the ratio (2.28) as the expression

$$u_L \doteqdot U_L(p), \quad i_L \doteqdot I_L(p).$$

Using the fourth property of Laplace transform (the image of the derivative), we find

$$U_L(p) = pLI_L(p) - Li_L(0).$$

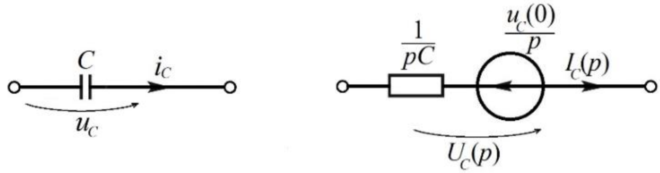
Here for a constant value $L$ the second property of Laplace transform (multiplication of the function $f(t)$ on the constant value) is taken into account also.

Thus, the operator image of the inductance $L$ is a serial connection of the operator resistance $pL$ and the voltage source, which electromotive force (e.m.f.) is $Li_L(0)$ and which in the direction coincides with the conditionally positive current direction (Fig .2.3,b).

Capacitance. For capacitance (Fig. 2.4,a) it is possible to write the relation between instantaneous values of current $i_C$ and voltage $u_C$.

$$i_C = C\frac{du_C}{dt},$$

$$u_C = \frac{1}{\mathrm{C}} \int_{-\infty}^{t} i_C dt = \frac{1}{C} \int_{-\infty}^{0} i_C dt + \frac{1}{\mathrm{C}} \int_{0}^{t} i_C dt = U_C(0) + \frac{1}{C} \int_{0}^{t} i_C dt.$$




a b

Fig. 2.4

In operational form it is written as

$$u_C \doteqdot U_C(p), \quad i_C \doteqdot I_C(p).$$

Using the fifth property of Laplace transform (the image of the integral) we find

$$u_C(p) = \frac{u_C(0)}{p} + \frac{1}{pC} I_C(p).$$

Here for the constant value $1/C$, the second property of Laplace transform (multiplication of the function $f(t)$ on the constant value) is applied, and for constant value $u_C(0)$ the first property of Laplace transform (image of constant value) is used. And for the sum of terms in expression (2.24) the third property of Laplace transform (linearity) is used.

Consequently, the operator image of the capacitance $C$ is the serial connection of the operator resistance $\frac{1}{pC}$ and the voltage source which e.m.f. is equal to $\frac{u_C(0)}{p}$ and is in the opposite direction to the conventionally positive current direction.

Thus, by all the elements of the electric circuit replacing with their operator images, one can obtain an equivalent operator scheme (EOS) of the circuit. So, for the circuit shown in Fig. 2.5,a we have such EOCS (Fig. 2.5,b).

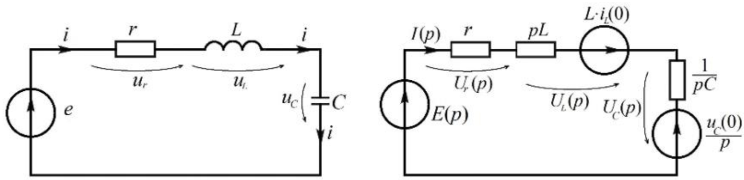
## 2.4. Ohm's and Kirchhoff's lows in operational form

***Ohm's low.*** For the scheme shown in the Fig. 2.5,a we can write according to the Ohm's law, where $e$ is e.m.f. of voltage source

$$u_r + u_L + u_C = e$$

or

$$ir + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} idt = e.$$




a b

Fig. 2.5

These equations may be represented in operational form (Fig.2.5,b) as

$$rI(p) = pLI(p) - Li_L(0) + \frac{1}{pC}I(p) + \frac{u_c(0)}{p} = E(p)$$

or

$$\left(r + pL + \frac{1}{pC}\right)I(p) = E(p) + Li_L(0) - \frac{u_c(0)}{p},$$

or

$$Z(p)I(p) = E_e(p), \quad I(p) = \frac{E_e(p)}{Z(p)}. \tag{2.29}$$

Where

$Z(p) = r + pL + \frac{1}{pC}$ – operator representation of electric resistance of circuit;

$E_e(p) = E_p(0) + Li_L(0) - \frac{u_c(0)}{p}$ – operator representation of equivalent e.m.f.

Expressions (2.29) are represented the Ohm's law in operator form.

*Kirchhoff's lows.* Using the property of linearity, one can write Kirchoff's law in image representation for currents and Kirchoff's law in image representation for voltagges.

Kirchoff's law for currents is

$$\sum_k i_k = 0 \doteqdot \sum_k I_k(p) = 0\,.$$

Kirchoff's law for voltage is

$$\sum_k u_k = 0 \doteqdot \sum_k U_k(p) = 0\,.$$

**2.5. Transient processes analysis with equivalent operation circuits**

The general procedure for transient processes calculation with using the EOS is following:

1) to determine the independent initial conditions (current in the inductance, voltage in the capacitor) with calculation of the pre-commutation circle;

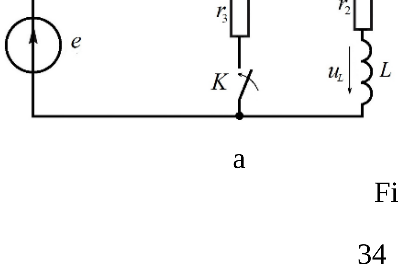
2) to make EOS according to the rules considered for the post-commutation circle;

3) to calculate EOS with any method of electric circuits calculation;

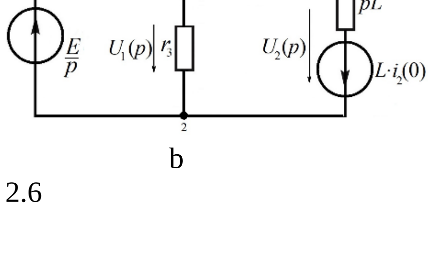
4) to find the originals of the searching values according to the resulting images with using the decomposition formula or the Table of originals and images.

**Example 2.1.**

To calculate the transient process with switch *K* closure in the circuit (Fig. 2.6,a) according to given EOS – method.




a




b

Fig. 2.6

*Solution*.

1. Independent initial conditions in before commutation conditions (switching (K is off) are define in according expression

$$i_2(0) = i_L(0) = \frac{E}{r_1 + r_2}$$

2. Equivalent operation circuit for after commutation circuit (switching $K$ is on) is shown in fig. 2.6, b.

In according voltage Kirchhoff low lets rout down system of node equation for nodes 1 and 2 can be rout down in form

$$\begin{bmatrix} g_1 + g_2 + g_3 & -g_2 \\ -g_2 & g_2 + \dfrac{1}{pL} \end{bmatrix} \begin{bmatrix} U_1(p) \\ U_2(p) \end{bmatrix} = \begin{bmatrix} \dfrac{E_1}{p} g_1 \\ -Li_2(0)\dfrac{1}{pL} \end{bmatrix}, \qquad (2.30)$$

where

$$g_1 = \frac{1}{r_1}, g_2 = \frac{1}{r_2}, g_3 = \frac{1}{r_3}.$$

3. From equation system (2.30) we get inductance voltage $U_2(p)$:

$$U_2(p) = \frac{\Delta_2}{\Delta}. \qquad (2.31)$$

Its evidence

$$\Delta = \begin{vmatrix} g_1 + g_2 + g_3 & -g_2 \\ -g_2 & g_2 + \dfrac{1}{pL} \end{vmatrix} = (g_1 + g_2 + g_3)\left(g_2 + \frac{1}{pL}\right) - g_2^2 =$$

$$(2.32)$$

$$= \frac{g_1 + g_2 + g_3 + (g_1 + g_3)g_2 pL}{pL};$$

$$\Delta_2 = \begin{vmatrix} g_1 + g_2 + g_3 & \dfrac{E_1}{p} g_1 \\ -g_2 & -Li_2(0)\dfrac{1}{pL} \end{vmatrix} = -(g_1 + g_2 + g_3)Li_2(0)\frac{1}{pL} +$$

$$(2.33)$$

$$+\frac{E_1}{p} g_1 g_2 = \frac{E g_1 g_2 - (g_1 + g_2 + g_3) i_2(0)}{p}$$

Now

$$U_2(p) = \frac{Eg_1g_2 - (g_1+g_2+g_3)i_2(0)}{(g_1+g_2+g_3)\frac{1}{L} + (g_1+g_3)g_2p} =$$

$$= \frac{b_0}{a_1p + a_0} = \frac{b_0}{a_1} \cdot \frac{1}{p + \frac{a_0}{a_1}} \ ,$$

where

$$a_0 = (g_1+g_2+g_3)\frac{1}{L} \ ;$$

$$a_1 = (g_1+g_3)g_2;$$

$$b_0 = E_xg_1g_2 - (g_1+g_2+g_3)i_2(0).$$

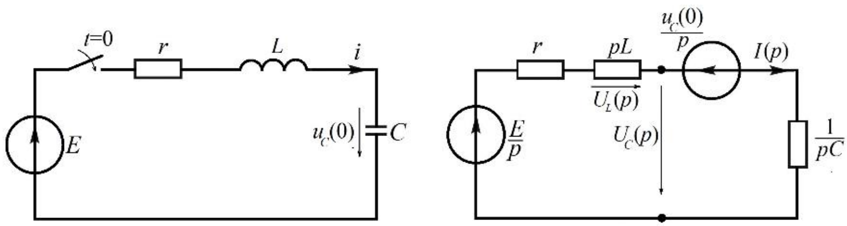
We get expression for original from the Table 2.1

$$U_2(p) \doteqdot u_2(t) = u_L = \frac{b_0}{a_1}e^{-\frac{a_0}{a_1}r}.$$

Here the first property of Laplace transform is used (image of constant value), second property (multiplication by a constant value) and seven property (argument shift) of Laplace transform are used also. Another values in scheme of Fig. 2.6,a can be find analogically.

## 2.6. Transient processes at turn on a non-branched circle of second order on a constant voltage

Let's analyze the process of switching the $rLC$-circle on a constant voltage (Fig. 7.2,a) according to the common calculation procedure. Let the capacitor $C$ is charged up to the voltage $u_c(0)$ in the pre-commutation circuit (switch $K$ turn on, Fig. 2.7,a), and current $i_L(0)$ in inductance $L$ equals zero. This is an independent initial condition.




a b

Fig. 2.6

According to the rules of operator image of elements obtaining, we compile EOS for the post-commutation circuit (switch $K$ is locked). The scheme is shown in Fig. 2, 7, b.

Let's calculate the scheme in Fig. 2.7,b.

It is obviously that

$$I(p) = \frac{\frac{E}{p} - \frac{u_c(0)}{p}}{r + pL + \frac{1}{pC}} = \frac{E - u_c(0)}{L} \frac{1}{p^2 + \frac{r}{L}p + \frac{1}{LC}} \, . \qquad (2.34)$$

Let's denote that $\delta = \frac{r}{2L}$ – attenuation coefficient, $\omega_0 = \frac{1}{\sqrt{LC}}$ – resonance frequency. Then from the expression (2.34) we get

$$I(p) = \frac{E - u_c(0)}{L} \frac{1}{p^2 + 2\delta p + \omega_0^2} =$$

$$= \frac{E - u_c(0)}{L} \frac{1}{(p - p_1)(p - p_2)}, \qquad (2.35)$$

where $p_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$ are roots of equation

$$p^2 + 2\delta p + \omega_0^2 = 0 \, . \qquad (2.36)$$

Depending on the values of the roots $p_{1,2}$ for circuit in Fig. 2.7, b the modes are distinguish on:

- aperiodic (at $\delta > \omega_0$); roots (2.36) – real and different, i.e. $\frac{r}{2L} > \frac{1}{\sqrt{LC}}$ or $r > 2\sqrt{\frac{L}{C}} = 2\rho = r_{cr}$, where $r_{cr} = 2\rho$ – critical resistance;
- critical (at $\delta = \omega_0$); roots (2.36) – real and same, $r = r_{cr}$;
- oscillatory (at $\delta < \omega_0$); roots (2.36) – complex and conjugated, $r < r_{cr}$.

Expressions for voltage in inductance and in capacitor have the form

$$U_L(p) = I(p)pL = \frac{E - u_c(0)}{L} pL \frac{1}{p^2 + 2\delta p + \omega_0^2} =$$

$$= [E - u_c(0)] \, \frac{p}{p^2 + 2\delta p + \omega_0^2}; \qquad (2.37)$$

$$U_C(p) = \frac{u_c(0)}{p} + I(p)\frac{1}{pC} = \frac{u_c(0)}{p} + \frac{E - u_c(0)}{LC}\frac{1}{p(p^2 + 2\delta p + \omega_0^2)} =$$
$$= \frac{u_c(0)}{p} + \omega_0^2[E - u_c(0)]\frac{1}{p(p^2 + 2\delta p + \omega_0^2)}. \qquad (2.38)$$

Expressions for the originals of the current are found by using the expansion formulas for the current images (2.34), (2.35) and (2.36).

Expression for curent $i(t)$ is defined according to the formula (2.17). According to the formula (2.34) we have: $F_1(p) = 1$; $F_2(p) = p^2 + 2\delta p + +\omega_0^2 = (p - p_1)(p - p_2)$, where $p_1$, $p_2$ are defined according to the formula (2.36);

$n = 2$;
$F_2'(p) = [(p - p_1)(p - p_2)]' = p - p_2 + p - p_1 = 2p - p_1 - p_2$;
$F_2'(p_1) = 2p_1 - p_1 - p_2 = p_1 - p_2$;
$F_2'(p_2) = 2p_2 - p_1 - p_2 = p_2 - p_1$.

So,

$$i(t) = \frac{E - u_c(0)}{L}\left(\frac{1}{p_1 - p_2}e^{p_1 t} + \frac{1}{p_2 - p_1}e^{p_2 t}\right) =$$
$$= \frac{E - u_c(0)}{(p_1 - p_2)L}(e^{p_1 t} - e^{p_2 t}). \qquad (2.39)$$

Expression for the original of inductance voltage $u_L(t)$ is defind according to the formula (2.17) also. And according to formula 1 (2.37) we have

$\mathrm{F_1(p) = p; F_2(p) = p^2 + 2\delta p + \omega_0^2; F_2'(p) = 2p - p_1 - p_2;}$
$\mathrm{F_1(p_1) = p_1;\ F_1(p_2) = p_2;\ F_2'(p_1) = p_1 - p_2;\ F_2'(p_2) = p_2 - p_1.}$

Then

$$u_L(t) = [E - u_c(0)]\left(\frac{p_1}{p_1 - p_2}e^{p_1 t} + \frac{p_2}{p_2 - p_1}e^{p_2 t}\right) =$$
$$= \frac{E - u_c(0)}{p_1 - p_2}(p_1 e^{p_1 t} - p_2 e^{p_2 t}).$$

To determine the original of capacity voltage $u_C(t)$ with taking into account that the denominator of the second term in formula (2.38) has a root equals to zero and by using the decomposition formula (2.18), according to which for the second term we obtain the following expression:

$F_1(p) = 1; F_3(p) = p^2 + 2\delta p + \omega_0^2; F_3'(p) = 2p - p_1 - p_2;$
$F_1(0) = 1; F_3(0) = \omega_0^2; F_3'(p_1) = p_1 - p_2; F_3'(p_2) = p_2 - p_1.$

Because $\frac{u_c(0)}{p} = u_c(0)$, then

$$u_c(t) = u_c(0) + \omega_0^2[E - u_c(0)] \times$$
$$\times \left[\frac{1}{\omega_0^2} + \frac{1}{p_1(p_1 - p_2)}e^{p_1 t} + \frac{1}{p_{2(}p_2 - p_1)}e^{p_2 t}\right] =$$
$$= u_c(0) + \omega_0^2[E - u_c(0)]\left[\frac{1}{\omega_0^2} + \frac{1}{p_1 - p_2}\left(\frac{1}{p_1}e^{p_1 t} - \frac{1}{p_2}e^{p_2 t}\right)\right] =$$
$$= u_c(0) + \omega_0^2[E - u_c(0)]\left[\frac{1}{\omega_0^2} + \frac{1}{\omega_0^2(p_1 - p_{2)}}(p_2 e^{p_1 t} - p_1 e^{p_2 t})\right] =$$
$$= u_c(0) + E - u_c(0) + \frac{E - u_c(0)}{p_1 - p_2}(p_2 e^{p_1 t} - p_1 e^{p_2 t}) =$$
$$E + \frac{E - u_c(0)}{p_1 - p_2}(p_2 e^{p_1 t} - p_1 e^{p_2 t})\,. \quad (2.40)$$

Here it is taken into account that

$$p_1 p_2 = \left(-\delta + \sqrt{\delta^2 - \omega_0^2}\right)\left(-\delta - \sqrt{\delta^2 - \omega_0^2}\right) = \omega_0^2\,.$$

## 2.7. Analysis of transient processes by the second order circuit turn on with constant voltage

For the electric scheme in Fig. 2.7, a the such processes are distinguished.

1. At $E = 0$ it is a free process in $rLC$ electric circuit, at which the capacitor charged to voltage $u_c(0)$ is completely discharged and then

$$i(t) = \frac{u_c(0)}{(p_2 - p_1)L}(e^{p_1 t} - e^{p_2 t}); \quad (2.41)$$
$$u_L(t) = \frac{u_c(0)}{(p_2 - p_1)}(p_1 e^{p_1 t} - p_2 e^{p_2 t}); \quad (2.42)$$
$$u_C(t) = \frac{u_c(0)}{p_2 - p_1}(p_2 e^{p_1 t} - p_1 e^{p_2 t}); \quad (2.43)$$

Let's consider the possible modes.

*Aperiodic mode* ($\delta > \omega_0$). Graphics of functions $i(t)$, $u_L(t)$, $u_C(t)$ are presented in Fig. 2.8.a,b,c respectively, where $|p_2| > |p_1|$. And $p_1 < 0,\ p_2 < 0,$ that is the roots are valid, different and negative.

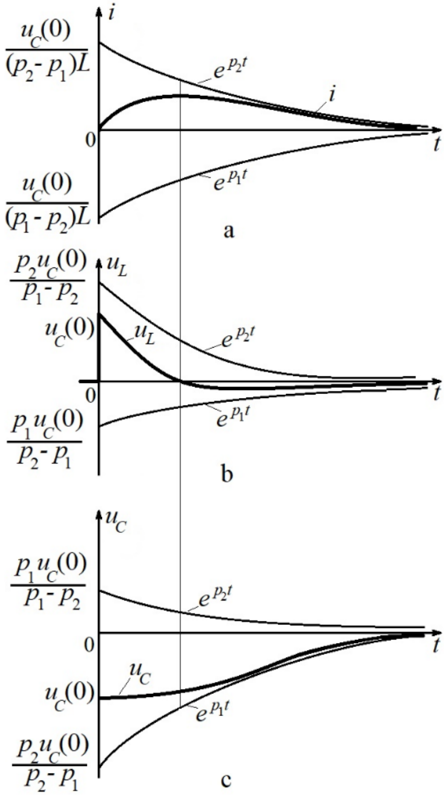



Fig.2.8

In Fig. 2.8.c $u_C(0) < 0$. From Fig.2.8 it follows that the current $i(t)$ increases from zero to the maximum value, and then falls to zero. The voltage $u_L(t)$ increases with jump at a switching moment $t = 0$ from zero value to $u_C(0)$ and then it decreases, passes through zero (when the current $i(t)$ reaches the maximum value), and then it becomes negative, and then grows to the maximum value and falls to zero value. Voltage $u_C(t)$ gradually decreases from value $u_C(0)$ to zerow value, the capacitor is discharged and the transient process is over.

*Oscillatory mode* ($\delta < \omega_0$). The roots $p_1$, $p_2$ are complex and conjugated:

$$p_{1,2} = -\delta \pm j\sqrt{\omega_0^2 - \delta^2} = -\delta \pm j\omega_{fre} = -\omega_0 e^{+ja}, \quad (2.44)$$

where

$$\omega_{fre} = \sqrt{\omega_0^2 - \delta^2};\ \alpha = \text{arctg}\frac{\omega_{fre}}{\delta}\ . \quad (2.45)$$

Then, according to the expression (2.41) and taking into account the relation (2.44) we have

$$i(t) = \frac{u_c(0)}{(-\delta - j\omega_{fre} + \delta - j\omega_{fre})L} \cdot \left[e^{(-\delta + j\omega_{fre})} - e^{(-\delta - j\omega_{fre})}\right] =$$

$$= -\frac{u_c(0)}{\omega_{fre} L} e^{-\delta t} \frac{e^{j\omega_{fre}t} - e^{-j\omega_{fre}t}}{2j} = -\frac{u_c(0)}{\omega_{fre}L} e^{-\delta t} \sin \omega_{fre} t. \quad (2.46)$$

Similarly, according to the expressions (2.42) and (2.43) and taking into account the relations (2.44) and (2.45) we have

$$u_L(t) = \frac{\omega_0}{\omega_{fre}} u_c(0) e^{-\delta t} \sin(\omega_{fre} t - \alpha); \quad (2.47)$$

$$u_c(t) = \frac{\omega_0}{\omega_{fre}} u_c(0) e^{-\delta t} \sin(\omega_{fre} t + \alpha); \quad (2.48)$$

According to formula (2.46) the plot of current change was constructed $i(t)$ (Fig. 2.9). The curent $i(t)$ varies according to the sinusoidal law, and the amplitude of the current drops by exponential law. Exponent $\frac{u_c(0)}{\omega_{fre}L} e^{-\delta t}$ is the bypass amplitude of the sinusoidal curve.

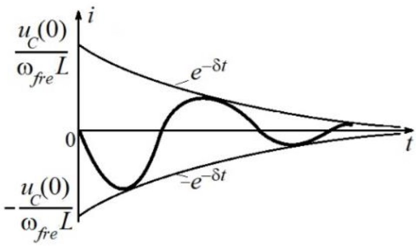



Fig. 2.9

2. At $u_c(0) = 0$ it is a turn on mode of *rLC*-curcuit with an uncharged capacitor on constant voltage. At the same time from the expressions (2.31) and (2.33) we have

$$i(t) = \frac{E}{(p_1 - p_2)L}(e^{p_1 t} - e^{p_2 t}); \tag{2.49}$$

$$u_L(t) = \frac{E}{p_1 - p_2}(p_1 e^{p_1 t} - p_2 e^{p_2 t}); \tag{2.50}$$

$$u_C(t) = E + \frac{E}{p_1 - p_2}(p_2 e^{p_1 t} - p_1 e^{p_2 t}). \tag{2.51}$$

Let's consider the possible modes.

*Аперіодичний режим* ($\delta > \omega_0$). The plots of fuctions $i(t)$, $u_L(t)$ and $u_C(t)$ are depicted in Fig.2.10. The capacitor $C$ (Fig.2.7,*a*) aperiodically charged from zero voltage to voltage value $E$. Current $i(t)$ in circuit is growing from zerou value up to maximum value (at the maximum speed of voltage $u_C(t)$ change). Voltage $u_L(t)$ in inductance at switching moment $t = 0$ grows with jump from zero value to $E$ value, then begins to decrease and passes through the zero value (at maximum current value $i(t)$); then becomes negative and increases to the maximum negative value (at the maximum speed of the current $i(t)$ change) and then goes down to zero.

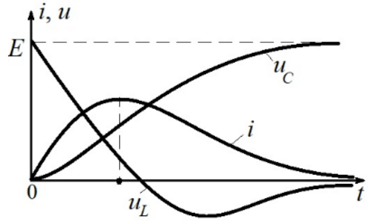



Fig.2.10

*Oscillatory mode* ($\delta < \omega_0$). At this mode similar to the formulas (2.46) - (2.48) from the equations (2.49) - (2.51) we obtain

$$i(t) = \frac{E}{\omega_{fre} L} e^{-\delta t} \sin \omega_{fre}(t); \tag{2.52}$$

$$u_L(t) = -\frac{\omega_0}{\omega_{fre}} E e^{-\delta t} \sin(\omega_{fre} t - \alpha); \tag{2.53}$$

$$u_C(t) = E - \frac{\omega_0}{\omega_{fre}} E e^{-\delta t} \sin(\omega_{fre} t + \alpha). \tag{2.54}$$

In the Fig. 2.11 the curves of current $i(t)$ and voltage $u_C(t)$ change according to formulas (2.52)-(2.54) are presented. Here current $i(t)$

changes as shown in Fig. 2.9. Voltage $u_C(t)$ according to oscillatory law and tends to source voltage *E*. If $t = \frac{T_{fre}}{2}$ and $\omega_0 \approx \omega_{fre}$, $\delta \ll \omega_{fre}$, namely $\alpha \approx \frac{\pi}{2}$, then expression becomes as

$$u_C\left(\frac{T_{fre}}{2}\right) = E + Ee^{-\delta\frac{T_{fre}}{2}} \approx 2E.$$

So, voltage at capacitor can be reached almost in twice more than source voltage *E*.

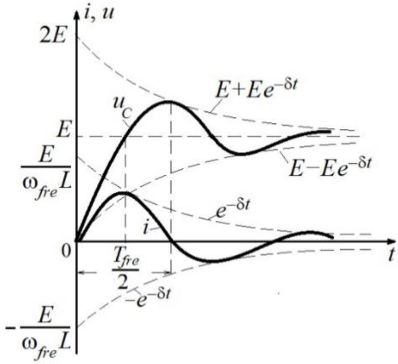



Fig. 2.11

## 2.8. Parameters of free oscillations

The oscillation process occurring in the *rLC*-circle is characterized by the following parameters:

1) the time constant of oscillation process

$$\tau = \frac{1}{\delta} = \frac{2L}{r},$$

this is the time at which the ordinate of the oscillatory amplitude of free oscillations decreases in *e* times (τ does not depend on capacity $C$);

2) duration of the oscillation process, namely, the transition time $t_{tr}$ for which amplitude of free oscillations $I_m$ decreases in 100 times. It is determined from the ratio

$$I_m e^{-\delta t_{tr}} = 0{,}01 I_m,$$

where

$$t_{tr} = \frac{\ln 100}{\delta} = \frac{4.6}{\delta} = 4.6\tau \,;$$

3) rate of the oscillatory process, namely, the rate of attenuation of free oscillations, which is determined by the ratio of amplitudes of oscillations at $t_1$ and $t_1 + T_{\text{в}}$ moments of time:

$$\frac{i(t_1)}{i(t_1 + T_{fre})} = \frac{I_m e^{-\delta t_1}}{I_m e^{-\delta(t_1 + T_{fre})}} = e^{\delta T_{\text{в}}} \,.$$

The quantity $\theta = \delta \cdot T_{fre} = \frac{2\pi\delta}{\omega_{fre}}$ is called as logarithmic attenuation decrement. For high-quality $rLC$-circuit ($\delta \ll \omega_0$) because $\omega_{fre} \approx \omega_0$ we have

$$\theta = \delta T_0 = \frac{2\pi\delta}{\omega_0} = \frac{\pi}{Q} \,,$$

where $Q$ is quality or $Q$ -factor of $rLC$-circuit

$$Q = \frac{\rho}{r} = \sqrt{\frac{L}{C}} \cdot \frac{1}{r} = \frac{L}{r\sqrt{LC}} = \frac{2L}{2r\sqrt{LC}} = \frac{\omega_0}{2\delta} \,.$$

## 2.9. Particularity of transient processes calculation by harmonic influences

Transient processes calculation by harmonic influences can be carry out by the straightly transfer from harmonic values to their Laplace images (table 2.1). But more expediency is preliminary transformation harmonic values into complex values and the next direct Laplace transforms. Both theirs modes are shown on structure diagram (fig. 2.12).

Reverse transfer from images to originals also carry out in two stages:

1) at first through known operational images the complex of instantaneous meaning value of originals,

2) and afterwards transfer from complex values to harmonic functions. Correspond diagram is shown in fig, 2.13.

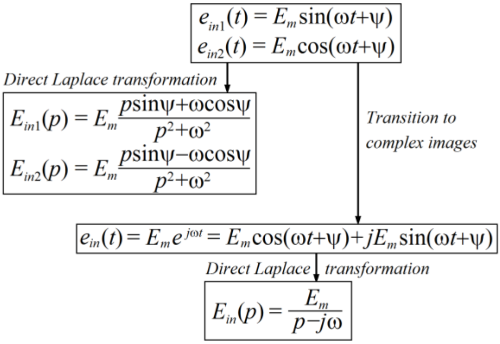



Fig. 2.12

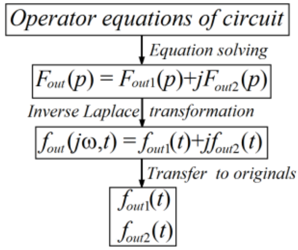



Fig. 2.13

**Example 2.2**

Calculate current $i(t)$ and voltage across capacitance $u_C(t)$ by switching *rC*-circuit to harmonic voltage $e(t) = E_m\cos(\omega t + \psi)$ in circuit fig.2.14.

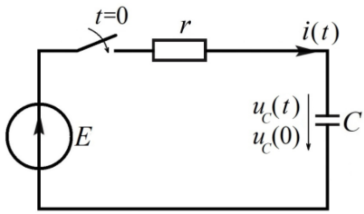



Fig. 2.14

*Solution.*

1. Let's considered solution by direct application Laplace transforms.

Let's find independent initial conditions in before commutation circuit.

Let's assume $u_C(0) = 0$

Let's compile equivalent operation circuit (EOC) for the after commutation regime, substituting EMF $e(t)$ for

$$L[e(t)] = E(p) = E_m \frac{p\cos\psi - \omega\sin\psi}{p^2 + \omega^2}. \qquad (2.55)$$

We compile EOC in fig. 2.15.

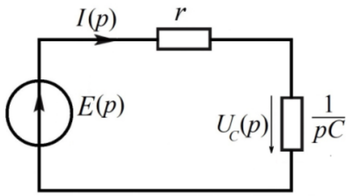



Fig. 2.15

Using voltage Kirchhoff's low (VKL), we get

$$I(p)(r+\frac{1}{pC})-E(p)=0. \qquad (2.56)$$

Solution of equation (2.56) gives

$$I(p)=\frac{E_m}{r}\frac{p^2}{p^2+\omega^2}\frac{e^{p\frac{\psi}{\omega}}}{p+\frac{1}{rC}}=\frac{E_m}{r}\cos\psi\frac{p(p-\omega t d\psi)}{(p^2+\omega^2)(p+\frac{1}{rC})}; \qquad (2.57)$$

$$U_C(p)=I(p)\frac{1}{pC}=\frac{E_m}{rC}\frac{p}{p^2+\omega^2}\frac{e^{p\frac{\psi}{\omega}}}{p+\frac{1}{rC}}=$$

$$=\frac{E_m}{rC}\cos\psi\frac{p-\omega t d\psi}{(p^2+\omega^2)\left(p+\frac{1}{rC}\right)}. \qquad (2.58)$$

Let's define originals of current and voltages, using expansion formula and property phase lag (2.7)

$$i(t)=\frac{E_m}{r}\frac{1}{\sqrt{(\omega rC)^2+1}}\times \qquad (2.59)$$

$$\times\left[\boldsymbol{\omega} rC\cos(\omega t+\psi+\alpha)-e^{-\frac{t}{rC}}\sin(\psi+\alpha)\right];$$

$$u_C(t)=\frac{E_m}{\sqrt{(\omega rC)^2+1}}\times \qquad (2.60)$$

$$\times\left[\sin(\omega t+\psi+\alpha)-e^{-\frac{t}{rC}}\sin(\psi+\alpha)\right],$$

where $\alpha=\mathrm{arctg}\frac{1}{\omega rC}$.

2. Let's considered solution by double transformation into complex and operation forms.

Let's substitute in after commutation circuit all elements by complex values

$$e(t) = E_m e^{j(\omega t+\psi)} = E_m e^{j\psi} = \dot{E}_m e^{j\omega t}; Z_r = r; Z_C = \frac{1}{j\omega C}. \quad (2.61)$$

Expression (2.61) can be represented in operational form

$$E(p) = \frac{E_m}{p - j\omega}; Z_r(p) = r; Z_C(p) = \frac{1}{pC}. \quad (2.62)$$

Using (2.57), (2.58), we get operation current and voltage

$$I(p) = \frac{E_m}{r} \frac{p}{(p - j\omega)\left(p + \frac{1}{rC}\right)}; \quad (2.63)$$

$$U_C(p) = \frac{E_m}{rC} \frac{1}{(p - j\omega)\left(p + \frac{1}{rC}\right)}. \quad (2.64)$$

Expressions for operational images of current (2.63) and voltage (2.64) are considerable simplily, then (2.57), (2.58). Originals of expressions (2.67), (2.68) gives complex instantaneous meaning current and voltage

$$I_m(t) = \frac{E_m}{r}\left(\frac{j\omega}{j\omega + \frac{1}{rC}} e^{j\omega t} + \frac{-\frac{1}{rC}}{-\frac{1}{rC} - j\omega} e^{-\frac{t}{rC}}\right) =$$

$$= \frac{rCE_m}{r(j\omega rC + 1)} \frac{j\omega rCe^{j\omega t} + e^{-\frac{t}{rC}}}{rC} = \quad (2.65)$$

$$= \frac{E_m}{r} \frac{1}{\sqrt{(\omega rC)^2 + 1}}\left[\omega rCe^{j\left(\omega t + \psi + \frac{\pi}{2} - a_{1)}\right)} + e^{-\frac{t}{rC}} e^{j(\psi - a_1)}\right];$$

$$U_m(t) = \frac{E_m}{rC}\left[\frac{1}{j\omega + \frac{1}{rC}} e^{j\omega t} + \frac{1}{-j\omega - \frac{1}{rC}} e^{-\frac{t}{rC}}\right] = \quad (2.66)$$

$$= \frac{E_m}{\sqrt{(\omega rC)^2 + 1}}[e^{j(\omega t + \psi - a_{1)}} - e^{-\frac{t}{rC}} e^{j(\psi - a_1)}],$$

where

$$a_1 = \mathrm{arctg}\, \omega rC = \frac{\pi}{2} - \mathrm{arctg} \frac{1}{\omega rC} = \frac{\pi}{2} - \alpha\,.$$

We pass from complex values to real form

$$i(t) = \frac{E_m}{r} \frac{1}{\sqrt{(\omega rC)^2 + 1}} \times \quad (2.67)$$

$$\times \left[\omega rC\cos(\omega t + \psi + \alpha) - e^{-\frac{t}{rC}} \sin(\psi + a_1)\right],$$

$$u_C(t) = \frac{E_m}{\sqrt{(\omega r C)^2 + 1}} \times \qquad (2.68)$$

$$\times \left[\sin(\omega t + \psi + \alpha) - e^{-\frac{t}{rC}} \sin(\psi + a_1)\right],$$

what is coincided with expressions (2.59), (2.60).

**Problem 2.1.**

Calculate and analyze transient processes in a given linear circuit of the second order which source of constant EMF (fig. P.1.10.by operational method.

*Solution.*

1. Let's find independent initial conditions - induction current $i_2(0)$, voltage across capacitance $u_C(0)$ – for before commutation circuit in Fig. P.2.1 (see Fig. P.1.11).

$$i_2(0) = i_L(0) = \frac{E}{r_2}, \;\; u_C(0) = 0.$$

2. Let's compile equivalent operational circuit for after commutation network (Fig. P.2.1).

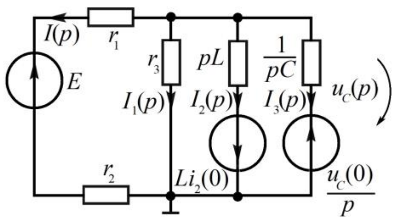



Fig. P.2.1

3. Let's calculate circuit in Fig. P.2.1, using node voltage method (NVM).

For the node 1 we get.

$$\left(\frac{1}{r_1 + r_2} + \frac{1}{r_3} + \frac{1}{pL} + pC\right) U_C(p) = \frac{\frac{E}{p}}{r_1 + r_2} - \frac{Li_2(0)}{pL} + \frac{\frac{u_C(0)}{p}}{\frac{1}{pC}};$$

where from

$$I_1(p) = \frac{U_C(p)}{r_3} = \frac{\dfrac{\frac{E}{p}}{r_1 + r_2} - \dfrac{Li_2(0)}{pL} + \dfrac{\frac{u_C(0)}{p}}{\frac{1}{pC}}}{\left(\dfrac{1}{r_1 + r_2} + \dfrac{1}{r_3} + \dfrac{1}{pL} + pC\right) r_3}.$$

After manipulation we get, using earlier received designation $\delta$ and $\omega_0$

$$I_1(p) = \frac{-Er_1}{r_2 r_3 C(r_1 + r_2)} \cdot \frac{1}{p^2 + 2\delta p + \omega_0^2}. \qquad \text{(P.2.1)}$$

4. Let's find original for (P.2.1), using expansion formula

$$f(t) = \sum_{k=1}^{n} \frac{F_1(P_k)}{F_2'(P_k)} e^{P_k t}. \qquad \text{(P.2.2)}$$

Let's represent (P.2.1) in form

$$I_1(p) = \frac{-Er_1}{r_2 r_3 C(r_1 + r_2)} \cdot \frac{1}{(p - p_1)(p - p_2)}.$$

where $p_1$, $p_2$ are defined from (P.1.28)

$F_2(p) = (p - p_1)(p - p_2)$; $\quad F_2'(p) = 2p - p_1 - p_2$.

Then

$$i_1(t) = \frac{-Er_1}{r_2 r_3 C(r_1 + r_2)} \left(\frac{1}{(p_1 - p_2)} e^{P_1 t} + \frac{1}{(p_2 - p_1)} e^{P_2 t}\right) = \qquad \text{(P.2.3)}$$

$$= \frac{Er_1}{r_2 r_3 C(r_1 + r_2)(p_2 - p_1)} (e^{P_1 t} - e^{P_2 t}).$$

It's shown current $i_1(t)$ coincide with current (P.1.40).

We calculate ruts $p_1$, $p_2$, use (P.1.28)

$$\tau = \frac{(r_1 + r_2) r_3 C}{r_1 + r_2 + r_3} = \frac{(10 + 10) 4 \cdot 10 \cdot 10^{-6}}{10 + 10 + 4} = 33{,}33 \cdot 10^{-5} \text{c};$$

$$\delta = \frac{1}{2\tau} = \frac{1}{2 \cdot 33{,}33 \cdot 10^{-5}} = 15 \cdot 10^3;$$

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{1 \cdot 10^{-3} \cdot 10 \cdot 10^{-6}} = 10^8 \text{c}^{-2};$$

$$p_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2} = -15 \cdot 10^3 \pm \sqrt{(15 \cdot 10^3)^2 - 10^8} =$$

$$= -15 \cdot 10^3 \pm 11{,}18 \cdot 10^3;$$

$$p_1 = -15 \cdot 10^3 + 11{,}18 \cdot 10^3 = -3{,}82 \cdot 10^3 \text{c}^{-1};$$

$p_2 = -15 \cdot 10^3 - 11{,}18 \cdot 10^3 = -26{,}18 \cdot 10^3 \text{c}^{-1}$;

Let's substitute meaning of parameters into expression (P.1.28) for $i_1(t)$

$$i_1(t) = \frac{100 \cdot 10}{10 \cdot 4(10+10) \cdot 10 \cdot 10^{-5}(-26{,}18 \cdot 10^3 + 3{,}82 \cdot 10^3)} \times \tag{P.2.4}$$

$$\times \left(e^{-3{,}82 \cdot 10^3 t} - e^{-26{,}18 \cdot 10^3 t}\right) = -59\left(e^{-3{,}82 \cdot 10^3 t} - e^{-26{,}18 \cdot 10^3 t}\right)\text{A}.$$

Let's designate

$$\tau_1 = \left|\frac{1}{p_1}\right| = \frac{1}{3{,}82 \cdot 10^3} = 0{,}262 \cdot 10^{-3} \text{c};$$

$$\tau_2 = \left|\frac{1}{p_2}\right| = \frac{1}{26{,}18 \cdot 10^3} = 0{,}038 \cdot 10^{-3} \text{c};$$

Rerate (P.2.4) in form

$$i_1(t) = -5{,}9\left(e^{-\frac{t}{0{,}262 \cdot 10^{-3}}} - e^{-\frac{t}{0{,}038 \cdot 10^{-3}}}\right) \tag{P.2.5}$$

Using (P.2.5), construct graphic of current $i_1(t)$ (Fig. P.2.2)

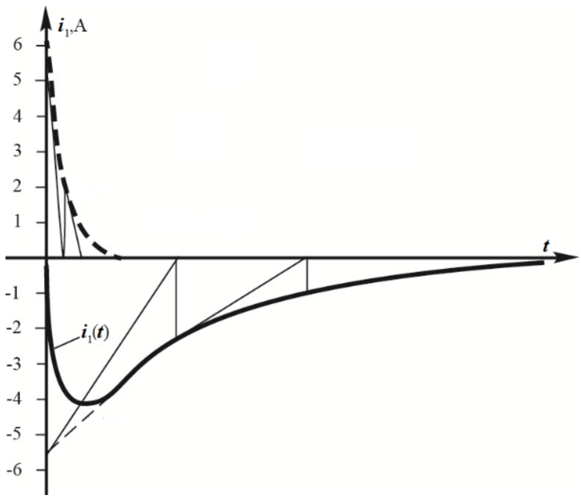



Fig. P.2.2

## Methodic instruction

Beginnings study section “Operational method of transient processes analyses”, necessary acquaintance from direct Laplace transform and its properties, give grant attention to expansion formula, as basic means of finding original from operation image.

After that to study methods of equivalent operation circuit construction, master the common order of operational method of transient processes calculation. For example it is expediency to considered transient processes in second order circuit.

Literature: [1] – [5]; [9] – [11].

## Questions for self checking

1. What is essence of operational method of transient processes analyses?
2. Written down expressions for direct and reverse Laplace transforms.
3. Formulate properties direct Laplace transform.
4. Explain expressions of expansions formula.
5. Compile equivalent operational circuit for the network of first and second order.
6. Give common order of transient processes with help equivalent operational circuit.
7. What is peculiarity operational transient processes calculation by harmonic influences?

# 3. CIRCUIT OPERATIONAL FUNCTIONS

## 3.1. Notion of circuit operational function

In the electric circuit the relation of the output value $x_{out}$ (reaction) with the input value $x_{in}$ (action) in the general case is presented as

$$a_m \frac{d^m x_{in}}{dt^m} + a_{m-1} \frac{d^{m-1} x_{out}}{dt^{m-1}} + \ ... + a_1 \frac{d x_{out}}{dt} + a_0 x_{out} =$$
$$= b_n \frac{d^n x_{in}}{dt^n} + b_{n-1} \frac{d^{n-1} x_{in}}{dt^{n-1}} + \ ... + b_1 \frac{d x_{in}}{dt} + b_0 x_{in} \ , \qquad (3.1)$$

where $a_0$, $a_1$, …, $a_m$, $b_0$, $b_1$, …, $b_n$ are the real coefficients which are determined by the electric circuit scheme and parameters of its elements.

If $x_{in} = X_{in}(p)$, $x_{out} = X_{out}(p)$, then in the operat form from the relation (3.1) we can write:

$$(a_m p^m + a_{m-1} p^{m-1} + \ ... \ + a_1 p + a_0) X_{out}(p) =$$
$$= (b_n p^n + b_{n-1} p^{n-1} + \ ... \ + b_1 p + b_0) X_{in}(p).$$

The operational function of electric circuit (OFC) $K(p)$ is the ratio of the image of the output value to the image of the input value at zero initial conditions:

$$K(p) = \frac{X_{out}(p)}{X_{in}(p)} = \qquad (3.2)$$
$$= \frac{b_n p^n + b_{n-1} p^{n-1} + \ ... \ + b_1 p + b_0}{a_m p^m + a_{m-1} p^{m-1} + \ ... \ + a_1 p + a_0} = \frac{N(p)}{M(p)}.$$

Here $n < m$ and OFC is the rational fraction. The polynomial $N(p)$ has roots $p_{01}$, $p_{02}$, …, $p_{0n}$ , which are the zeros of function OFC. The polinomial $M(p)$ has roots $p_{p1}$, $p_{p2}$, …, $p_{pm}$ , which are the poles of function OFC. So,

$$K(p) = \frac{N(p)}{M(p)} = K \frac{(p - p_{01})(p - p_{02}) \ ... \ (p - p_{0n})}{(p - p_{p1})(p - p_{p2}) \ ... \ (p - p_{pm})}, \qquad (3.3)$$

where $K = \frac{b_n}{a_m}$.

That is the function OFK and hence the circuit itself are completely determined by the values of their zeros and poles on the complex plane.

The operational function of electric circuit OFC are used to describe the electric circuits without independent energy sources at zero initial conditions.

### 3.2. Variety of circuit operational function (COF)

Let's describe the scheme is presented in the Figure 3.1.

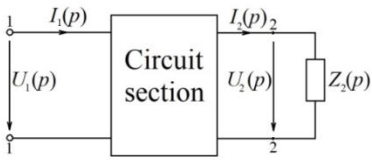



Fig. 3.1

The operational function of electric circuit is the ratio of operational currents or voltages of some element of electric circuit to the operational current or voltage in the input of th electric circuit.

There are input and transient OFK functions:

1) the *input OFK function* is ratio of the operational current or voltage in the input clamps of the electric circle. That is the operational input resistance $Z_{11}(p)$ and its conductivity $Y_{11}(p)$:

$$Z_{11}(p) = \frac{U_1(p)}{I_1(p)}; \;\; Y_{11}(p) = \frac{I_1(p)}{U_1(p)}.$$

So,

$$Z_{11}(p) = \frac{1}{Y_{11}(p)};$$

2) the *transient OFK function* is the ratio of operational currents or voltages in different clampes of electric circuit. That is the transient operational resistance $Z_{21}(p)$ and conductivity $Y_{21}(p)$, and

operational transient coefficients over current $K_{I21}(p)$ and voltage $K_{U21}(p)$:

$$Z_{21}(p)=\frac{U_2(p)}{I_1(p)};\;\; Y_{21}(p)=\frac{I_2(p)}{U_1(p)};$$
$$K_{I21}(p)=\frac{I_2(p)}{I_1(p)};\;\; K_{U21}(p)=\frac{U_2(p)}{U_1(p)},$$
$$Z_{21}(p)\neq\frac{1}{Y_{21}(p)}.$$

**Example 3. 1**

Define circuit operation function for the circuit (fig.E.3.2)

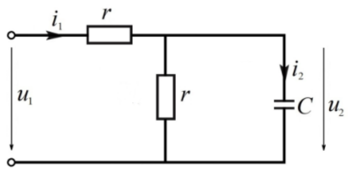



Fig. E.3.2

*Solution.*

Let's represent elements of the circuit in operation form

$$Z_{11}(p)=\frac{U_1(p)}{I_1(p)}=r+\frac{r}{rpC+1}=r\frac{rpc+2}{rpc+1};$$
$$Y_{11}(p)=\frac{I_1(p)}{U_1(p)}=\frac{1}{Z_{11}(p)}=\frac{1}{r}\cdot\frac{1+rpC}{2+rpC};$$
$$Z_{21}(p)=\frac{U_2(p)}{I_1(p)}=\frac{r}{1+rpC}=\frac{1}{C}\cdot\frac{1}{p+\frac{1}{rC}};$$
$$Y_{21}(p)=\frac{I_2(p)}{U_1(p)}=\frac{Y_{11}(p)}{1+rpC}=\frac{1}{r}\frac{p}{p+\frac{2}{rC}};$$
$$K_{I21}(p)=\frac{I_2(p)}{I_1(p)}=\frac{p}{p+\frac{1}{rC}};$$
$$K_{U21}(p)=\frac{U_2(p)}{U_1(p)}=\frac{r}{r^2pC+2r}=\frac{1}{rC}\cdot\frac{1}{p+\frac{2}{rC}}.$$

This example shows COF depends only on circuit structure and don't depends on input action.

**Problem 3.1.**

Calculate circuit operation function (transference admittance) for the given linear circuit of the second order (fig. P.3.1).

*Solution.*

1.Let's find circuit operation function (transference admittance).

$$Y_{21}(p)=\frac{I_2(p)}{U_1(p)}=\frac{\Delta_{12}}{\Delta_{11}}=Y_3(p) \qquad \text{(P.3.1)}$$

that is operation transference admittance

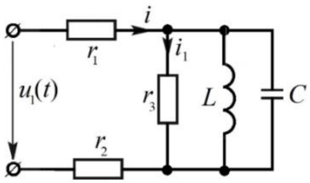



Fig. P.3.1

Let's compile equivalent operation circuit (fig.P.3.2).

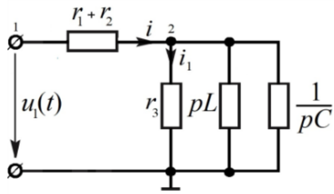



Fig. P.3.2

Designating basis node, rout down matrix node conductance (MNC) for the nodes 1, 2

$$\Delta=\begin{bmatrix} \frac{1}{r_1+r_2} & -\frac{1}{r_1+r_2} \\ -\frac{1}{r_1+r_2} & \frac{1}{r_1+r_2}+\frac{1}{r_3}+\frac{1}{pL}+pC \end{bmatrix}.$$

Where from

$$\Delta_{12}=\frac{1}{r_1+r_2};\ \Delta_{11}=\frac{1}{r_1+r_2}+\frac{1}{r_3}+\frac{1}{pL}+pC.$$

As $Y_3(p) = \frac{1}{r_3}$, then

$$Y_{21}(p) = \frac{1}{(r_1 + r_2)\left(\frac{1}{r_1 + r_2} + \frac{1}{r_3} + \frac{1}{pL} + pC\right)} \cdot \frac{1}{r_3} =$$

$$= \frac{(r_1 + r_2)r_3pL}{(r_1 + r_2)[r_3pL + (r_1 + r_2)pL + (r_1 + r_2)r_3 + (r_1 + r_2)r_3p^2LC]} \cdot \frac{1}{r_3}$$

$$= \frac{Lp}{(r_1 + r_2)r_3LCp^2 + (r_1 + r_2 + r_3)Lp + (r_1 + r_2)r_3} =$$

$$= \frac{1}{(r_1 + r_2)r_3C} \cdot \frac{p}{p^2 + \frac{r_1 + r_2 + r_3}{(r_1 + r_2)r_3C}p + \frac{1}{LC}}.$$

Using received designation, we get

$$Y_{21}(p) = \frac{1}{(r_1 + r_2)r_3C} \cdot \frac{p}{p^2 + 2\delta p + \omega_0^2}.$$

Circuit operation function can be transferred into circuit complex function ($K(j\omega)$) by substitution operator $p$ on image frequency $j\omega$

$$K(j\omega) = K(p)$$

For COF can be right

$$K(p) = K(\sigma + j\omega) = R(\sigma, \omega) + jX(\sigma, \omega) = K(\sigma, \omega)e^{j\varphi(\sigma,\omega)}, \sigma = 0,$$

where $R(\sigma, \omega)$, $X(\sigma, \omega)$ – real and image part of $K(\sigma + j\omega)$; $K(\sigma, \omega)$, $\varphi(\sigma, \omega)$ – module and argument of COF. We remind of $p = \sigma + j\omega$.

Components $R(\sigma, \omega)$, $X(\sigma, \omega)$, $K(\sigma, \omega)$, $\varphi(\sigma, \omega)$ of COF $K(p)$ are functions of two variables σ and ω. That is way their can by given surfaces. At $\sigma = 0$ components COF become components of circuit complex functions (CKF), that is way frequency characteristics.

Circuit operation functions of linear circuit with finite number elements can by always represent fractional – rational functions in form relations determinants $N(p)$ and $M(p)$ (see 3.3). Zeros and poles of COF can be real or in pairs conjugate complex numbers. If they are real numbers processes in the circuit are periodic, if they are complex numbers – oscillator characters. In ideal passive circuit with only *L*-,*C*-elements free oscillators their amplitudes aren’t decries, in real circuits amplitudes decries in time

### 3.3. Transient processes analyze by of circuit operational functions

Circuit operational functions are widely used for analyze processes in electrical circuits.

It is shown from (3.2)

$$X_{out}(p) = K(p)X_{in}(p). \tag{3.4}$$

So, the reaction of electric circuit $X_{out}(p)$ on the arbitrary input action $X_{in}(p)$ can be determined, if we know corresponding OFK function.

From expression (3.4) follows the following order of analysis:

1) make an equivalent circuit scheme of electric circuit;
2) to calculate the corresponding OFK function;
3) to find the operational image of input action $X_{in}(p)$;
4) to calculate the operational image of electric circuit action $X_{out}(p)$;
5) to determine the original of the found value $X_{out}(p)$ according to the tables or expansion formulas.

As an example, let's consider the passage of signals through an electric circuit with OFK function.

**Example 3.2.**

The impulse of exponential form is applied to the input of electrical circuit (fig. E.3.2)

$$U_{in}(t) = U_m e^{-\alpha t}. \tag{3.5}$$

Find current of capacitance $i_C(t)$.

*Solution.*

1. Image of the input action $U_{in}(t) = \frac{U_m}{p+\alpha}$.

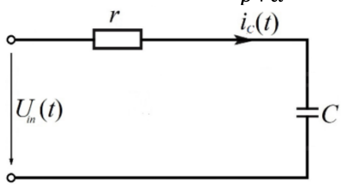



Fig. E.3.2

2.Image of the capacitance voltage

$$U_C(p) = \frac{U_m}{(p+\alpha)(2+prC)}.$$

3. Image of the capacitance current

$$I_C(p) = \frac{U_m rpC}{(p+\alpha)(rpC+2)}. \quad (3.6)$$

4. Characteristscal equations and its ruts

$$(p+\alpha)(rpC+2) = 0,\ p_1 = -\alpha,\ p_2 = -\frac{2}{rC}.$$

5. Original of capacitance current

$$i_C(t) = U_m \frac{\alpha e^{-\alpha t} - \frac{2e^{-\frac{2t}{rC}}}{rC}}{r\left(\alpha - \frac{2}{rC}\right)}). \quad (3.7)$$

**Example 3.3.**

Switch $K$ is closed and apply the constant voltage $u_{in}(t) = E$ to the input of the circuit (fig. 3.4). Define current $i_L(t)$ using circuit operational functions.

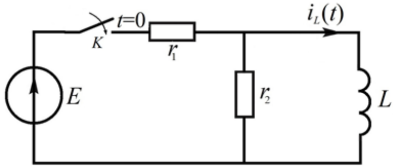



Fig. 3.4

*Solution.*

1. Image of input influence (table 2.1)

$$L[u_{in}(t)] = L(E) = U_{in}(p) = \frac{E}{p}. \quad (3.5)$$

2. Image of inductance current (table 2.1)

$$I_L(p) = \frac{Er_2}{(r_1+r_2)L} \cdot \frac{1}{p\left[p + \frac{r_1 r_2}{(r_1+r_2)L}\right]}. \quad (3.6.)$$

Or

$$I_L(p) = K\frac{1}{p(p-p_1)},$$

where

$$K=\frac{Er_2}{(r_1+r_2)L}, \quad p1=-\frac{r_1r_2}{(r_1+r_2)L}. \tag{3.7}$$

That is way original

$$i_L(t)=\frac{E}{r_1}\left[1-e^{-\frac{r_1r_2}{(r_1+r_2)L}t}\right]. \tag{3.8}$$

Input action is shown in fig. 3.5, graphic $i_L(t)$ – in fig.3.6

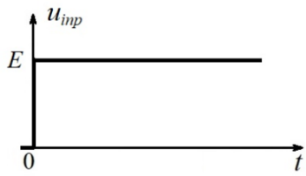



Fig. 3.5

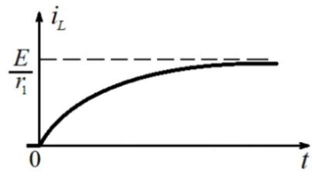



Fig. 3.6

**Example 3.4**

Calculate current $i(t)$ in the circuit (fig. 3.7) which is connected to voltage (fig. 3,8), using circuit operational function.

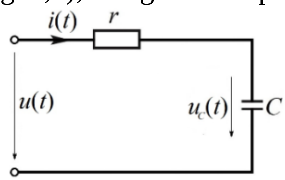



Fig.3.7

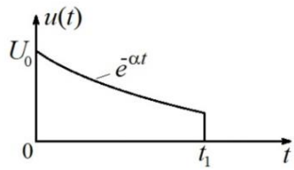



Fig.3.8

At interval 0-$t_1$ input voltage is changed exponentially

$$u(t)=U_0e^{-\alpha t}. \tag{3.9}$$

If reaction is current $i(t)$ and influence – voltage $u(t)$ then operation transfer admittance

$$Y(p)=\frac{I(p)}{U(p)}=\frac{U(p)}{r+\frac{1}{pC}}\cdot\frac{1}{U(p)}=\frac{pC}{rpC+1}=\frac{1}{r}\cdot\frac{p}{p+\frac{1}{rC}}. \tag{3.10}$$

Operation image of input influence can be founded through direct Laplace transform (2.1), changed upper bound of integration

$$U(p)=\int_0^{t_1}u(t)\,e^{-pt}dt=\int_0^{t_1}U_0e^{-\alpha t}\,e^{-pt}dt=$$

$$= U_0 \int_0^{t_1} e^{-(p+\alpha)t} dt = \frac{U_0}{p+\alpha}\left[1 - e^{-(p+\alpha)t_1}\right]. \qquad (3.11)$$

Now in accordance (3.10), (3.11) we get

$$I(p) = Y(p)U(p) = \frac{1}{r} \cdot \frac{p}{p + \frac{1}{rC}} \cdot \frac{U_0}{p+\alpha}\ \left[e^{-(p+\alpha)t_1}\right] =$$

$$= \frac{U_0}{r}\left[\frac{p}{\left(p+\frac{1}{rC}\right)(p+\alpha)} - \frac{p}{\left(p+\frac{1}{rC}\right)(p+\alpha)} e^{-(p+\alpha)t_1}\right]. \quad (3.12)$$

Original of image (3.12) is designation for every item separately.

For the first item in square brackets we get $\Phi 1(p) = \frac{p}{\left(p+\frac{1}{rC}\right)(p+\alpha)}$ in according expansion formula (2.17) we get

$F_1(p) = p$; $F_2(p) = (p + \frac{1}{rC})(p + \alpha)$; $n = 2$; $p_1 = -\frac{1}{rC}$; $p_2 = -\alpha$; $F_2'(p) = 2p + \frac{1}{rC} + \alpha$.

Then original

$$L^{-1}[\Phi_1(p)] = \frac{1}{\alpha - \frac{1}{rC}}\left(\alpha e^{-\alpha t} - \frac{1}{rC} e^{-\frac{t}{rC}}\right) = \Phi_1(t). \qquad (3.13)$$

For the second item in square brackets we get

$$\Phi_2(p) = \frac{p}{\left(p+\frac{1}{rC}\right)(p+\alpha)} e^{-(p+\alpha)t_1} = e^{-\alpha t_1}\Phi_1(p) e^{-pt_1}.$$

Were from in according phase lag (2.7) we get

$$\Phi_2(t) = e^{-\alpha t_1}\Phi_1(t - t_1) \cdot 1(t - t_1) =$$

$$= \frac{e^{-\alpha t_1}}{\alpha - \frac{1}{rC}}\left[\alpha e^{-\alpha(t - t_1)} - \frac{1}{rC} e^{-\frac{t - t1}{rC}}\right] \cdot 1(t - t1). \qquad (3.14)$$

Now, regarding (3.12) – (3.14), we get original current $i(t)$

$$i(t) = \frac{U_0}{r}[\Phi_1(t) - \Phi_2(t)] = \frac{U_0}{r}\left\{\frac{1}{\alpha - \frac{1}{rC}}\left[\alpha e^{-\alpha t_1} - \frac{1}{rC} e^{-\frac{t}{rC}}\right] - \right.$$

$$\left. - \frac{e^{-\alpha t_1}}{\alpha - \frac{1}{rC}}\left[\alpha e^{-\alpha(t - t_1)} - \frac{1}{rC} e^{-\frac{t - t_1}{rC}}\right] \cdot 1(t - t_1)\right\}.$$

If $t < t_1$, by $1(t - t_1) = 0$ we get

$$i(t) = \frac{U_0}{r}\frac{1}{\alpha-\frac{1}{rC}}\left(e^{-\alpha t_1} - \frac{1}{rC}e^{-\frac{t}{rC}}\right) - \frac{U_0}{r(1-\alpha rC)}\left(e^{-\frac{t}{rC}} - \alpha rCe^{-\alpha t}\right). \quad (3.15)$$

If $t \geq t_1$, by $1(t - t_1) = 1$ we get

$$i(t) = \frac{U_0}{r}\left\{\frac{1}{\alpha - \frac{1}{rC}}(e^{-\alpha t_1} - \frac{1}{rC}e^{-\frac{t}{rC}}) - \frac{e^{-\alpha t_1}}{\alpha - \frac{1}{rC}}\left[\alpha e^{-\alpha(t - t_1)} - \right.\right.$$

$$\left.\left. - \frac{1}{rC}e^{-\frac{t - t_1}{rC}}\right]\right\} = \frac{U_0}{r(1-\alpha rC)}\left(e^{-\frac{t}{rC}} - \alpha rCe^{-\alpha t}\right). \quad (3.16)$$

Where from, if $t = t_1$

$$i(t) = \frac{U_0}{r(1-\alpha rC)}\left[1 - e^{\left(\frac{t}{rC}-\alpha\right)t_1}\right]e^{-\frac{t_1}{rC}} =$$

$$= \frac{U_0}{r(1-\alpha rC)}\left(e^{-\frac{t_1}{rC}} - e^{-\alpha t_1}\right).$$

From (3.15) if $t \to 0$ we get $i(t) \to 0$.

Operate image of input action can be receive represented input voltage as difference of simplest actions

$$u(t) = u_1(t) - u_2(t),$$

where $u_1(t)$ can be expressed in according (3.9)

$$u_1(t) = U_0 e^{-\alpha t} \cdot 1(t), \quad (3.17)$$

and $u_2(t)$ is shifted at $t_1$ exponent

$$u_2(t) = U_0 e^{-\alpha t} e^{-\alpha(t - t_1)} \cdot 1(t - t_1). \quad (3.18)$$

From (3.17) and table 2.1 we get

$$U_1(p) = \frac{U_0}{p+\alpha},$$

then from (3.18) in accordance (2.7) we get

$$U_2(p) = \frac{U_0}{p+\alpha}\, e^{-\alpha t_1} e^{-pt_1}.$$

and as result

$$U(p) = U_1(p) - U_2(p) = \frac{U_0}{p+\alpha}\left[1 - e^{-(p+\alpha)t_1}\right],$$

what is coincide with (3.11).

**Problem 3.2**

Calculate and analyze of exponential video impulse passing in a given linear passive circuit of the second order (fig. P.3.1) with help circuit operational function: $u(t) = Ee^{-\alpha t}$, where α = $0.3|p_{min}|$, $p_{min}$ – lesser module root of characteristically equation. Find current $i_1(t)$.

*Solution.*

Operational image of input signal

$$U(p) = \frac{E}{p+\alpha}. \tag{P.3.2}$$

Then image of output current I1(p), using (P.3.2) and (P.3.1)

$$I_1(p) = U(p)Y_{21}(p) = E\frac{1}{p+\alpha}\cdot\frac{p}{(r_1+r_2)r_3C(p^2+2\delta p+\omega_0^2)} =$$

$$= \frac{E}{(r_1+r_2)r_3C}\cdot\frac{p}{(p-0.3p_1)(p-p_1)(p-p_2)}. \tag{P. 3.3}$$

Here $\alpha = 0{,}3|p_{min}| = -0{,}3\,p_1$

Let's find original of (P.3.3) by expansion theorem, using formula (P.2.2). Here

$$F_2(p) = (p-0{,}3p_1)(p-p_1)(p-p_2). \tag{P.3.4}$$

Then derivative of (P.3.4) will be rout down as

$F_2'(p) = (p-p_1)(p-p_2) + (p-0{,}3p_1)(2p-p_1-p_2)$.

Original from (P.3.4) is equal to

$$i_1(t) = \frac{E}{(r_1+r_2)r_3C}\left[\frac{0{,}3p_1}{-0{,}7p_1(0{,}3p_1-p_2)}e^{0{,}3p_1t} + \right.$$

$$\left. + \frac{p_1}{0{,}7p_1(p_1-p_2)}e^{p_1t} + \frac{p_2}{(p_2-0{,}3p_1)(p_1-p_2)}e^{p_2t}\right] =$$

$$= \frac{E}{(r_1+r_2)r_3C(p_1-p_2)}\left(0{,}429\frac{p_1-p_2}{p_2-0{,}3p_1}e^{0{,}3p_1t} + 1{,}429e^{p_1t} - \right.$$

$$\left. - \frac{p_2}{p_2-0{,}3p_1}e^{p_2t}\right). \tag{P.3.5}$$

Let's substitute into (P.3.5) numerate values. We get

$$i_1(t) = \frac{1}{(10+10)\cdot 4\cdot 10\cdot 10^{-6}(-3{,}82\cdot 10^3 + 26{,}18\cdot 10^3)} \times$$

$$\times\left[0{,}429\frac{(-3{,}82\cdot 10^3 + 26{,}18\cdot 10^3)\cdot e^{-0{,}3\cdot 3{,}82\cdot 10^3 t}}{-26{,}18\cdot 10^3 + 0{,}3\cdot 3{,}82\cdot 10^3} + \right.$$

$$\left. + 1{,}429\cdot e^{-3{,}82\cdot 10^{-3}t} - \frac{-26{,}18\cdot 10^3\cdot e^{-26{,}18\cdot 10^3 t}}{-26{,}18\cdot 10^3 + 0{,}3\cdot 3{,}82\cdot 10^3}\right] = \tag{P.3.6}$$

$$= -2{,}14\cdot e^{-\frac{t}{0{,}873\cdot 10^{-3}}} + 7{,}99\cdot e^{-\frac{t}{0{,}262\cdot 10^{-3}}} - 5{,}85\cdot e^{-\frac{t}{0{,}038\cdot 10^{-3}}}\text{A}.$$

Graphic $i(t)$ is shown in fig. P.3.3

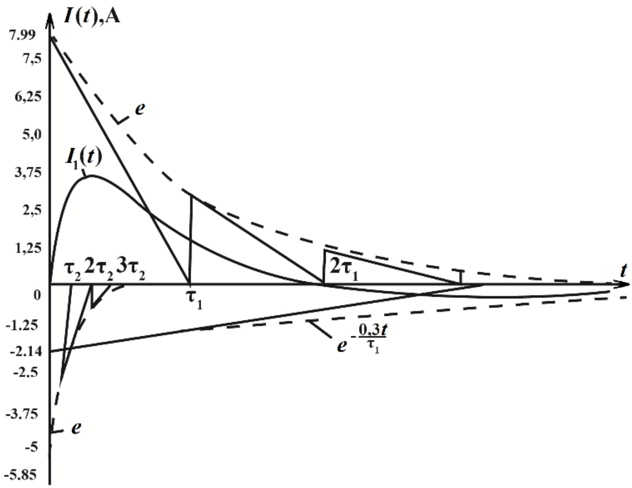



Fig. P.3.3

**Problem 3.3.**

Calculate and analyze of exponential video impulse passing in a given linear passive circuit of the second order (fig. P.3.1) with help Duhamel integral (time method) by input action $u(t) = Ee^{-\alpha t}$, where $\alpha = 0.3|p_{min}|$, $p_{min}$ – lesser module root of characteristically equation. Find current $i_1(t)$.

*Solution.*

Lets used the following form of convolution integral (Duhamel)

$$f_{out}(t) = \int_0^t f_{in}(t - \tau)\, a(\tau) d\tau. \qquad \text{(P.3.7)}$$

Here:

$$\begin{cases} f_{out}(t) = i_1(t); \\ h(t) = h_{Y_1}(t); \text{ if } t = 0 \;\; h_{Y_1}(t) = 0, \text{ according to (3.33);} \\ f_{in}(t) = u_1(t) = Ee^{-\alpha t}, \; f_{in}(t - \tau) = u_1(t - \tau) = Ee^{-\alpha(t-\tau)}; \\ a(t) = a_{Y_{21}}(t); \, a_{Y_{21}}(\tau) = \dfrac{1}{(r_1 + r_2) r_3 C (p_1 - p_2)} (p_1 e^{p_1 \tau} - p_2 e^{p_2 \tau}). \end{cases} \quad \text{(P.3.8)}$$

After substitution (P.3.8) into (P.3.7) we get

$$i_1(t) = \int_0^t Ee^{-\alpha(t-\tau)} \frac{1}{(r_1+r_2)r_3C(p_1-p_2)}(p_1e^{p_1\tau}-p_2e^{p_2\tau})d\tau =$$

$$= \frac{Ee^{-\alpha t}}{(r_1+r_2)r_3C(p_1-p_2)} \int_0^t e^{\alpha\tau}(p_1e^{p_1\tau}-p_2e^{p_2\tau})d\tau =$$

$$= \frac{Ee^{-\alpha t}}{(r_1+r_2)r_3C(p_1-p_2)} \left[p_1\int_0^t e^{(\alpha+p_1)\tau}d\tau - p_2\int_0^t e^{(\alpha+p_2)\tau}d\tau\right] =$$

$$= \frac{Ee^{-\alpha t}}{(r_1+r_2)r_3C(p_1-p_2)} \left[\frac{p_1}{\alpha+p_1}e^{(\alpha+p_1)\tau}\Big|_0^t - \frac{p_2}{\alpha+p_2}e^{(\alpha+p_2)\tau}\Big|_0^t\right] =$$

$$= \frac{Ee^{-\alpha t}}{(r_1+r_2)r_3C(p_1-p_2)}\left\{\frac{p_1}{\alpha+p_1}\left[e^{(\alpha+p_1)\mathrm{t}}-1\right] - \right.$$
$$\left. - \frac{p_2}{\alpha+p_2}\left[e^{(\alpha+p_2)\mathrm{t}}-1\right]\right\} =$$

$$= \frac{E}{(r_1+r_2)r_3C(p_1-p_2)}\left[\frac{p_1}{\alpha+p_1}(e^{p_1\mathrm{t}}-e^{-\alpha t}) - \right.$$
$$\left. - \frac{p_2}{\alpha+p_2}(e^{p_2\mathrm{t}}-e^{-\alpha t})\right] = \qquad \text{(P.3.9)}$$

$$= \frac{E}{(r_1+r_2)r_3C(p_1-p_2)}\left[\frac{p_1}{\alpha+p_1}e^{p_1\mathrm{t}} - \frac{p_2}{\alpha+p_2}e^{p_2\mathrm{t}} - \right.$$
$$\left. -\left(\frac{p_1}{\alpha+p_1} - \frac{p_2}{\alpha+p_2}\right)e^{-\alpha t}\right] =$$

$$= \frac{E}{(r_1+r_2)r_3C(p_1-p_2)}\left[\frac{p_1}{p_1-0{,}3p_1}e^{p_1\mathrm{t}} - \frac{p_2}{p_2-0{,}3p_1}e^{p_2\mathrm{t}} + \right.$$
$$\left. + \frac{(p_1-p_2)0{,}3p_1}{(p_1-0{,}3p_1)(p_2-0{,}3p_1)}e^{0{,}3p_1\mathrm{t}}\right] = \frac{E}{(r_1+r_2)r_3C(p_1-p_2)} \times$$
$$\times\left(1{,}429e^{p_1\mathrm{t}} - \frac{p_2}{p_2-0{,}3p_1}e^{p_2\mathrm{t}} + 0{,}429\frac{p_1-p_2}{p_2-0{,}3p_1}e^{0{,}3p_1\mathrm{t}}\right).$$

It is shown, (P.3.9) coincide with result by operation method. There for we get identical graphic (fig. 3.9).

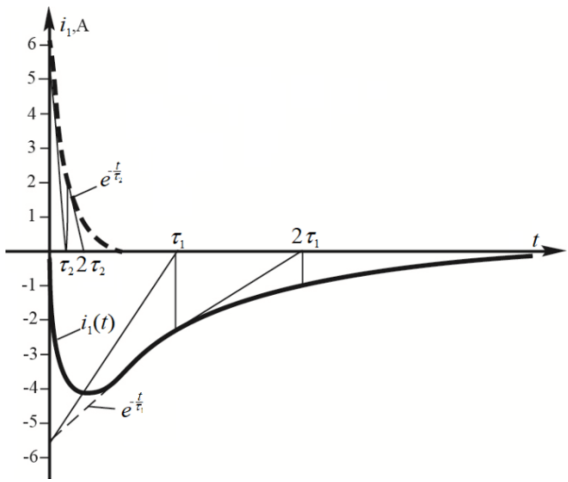



Fig. P.3.9

## Methodic instruction

Its necessary realize essence circuit operational function and there variety, master order of calculate and analyze transient processes with help circuit operational functions. Theoretical material is fixed by examples of calculations for pass signals of composite form through electrical circuits.

Literature: [1] - [4]; [9]; [14] - [16]

## Questions for self checking

1. What are circuit operational functions? What are varieties of them?
2. What is connected circuit operational function with circuit complex function?
3. Give an example of transient processes calculation with help circuit operational function.

# 4. METHOD OF CONVOLUTION INTEGRAL

## 4.1. Superposition method in transient processes theory

If to a linear electric circuit the complex action $x_{in}(t)$, which equals to the sum of simple input actions $x_{in,k}(t)$ is applied

$$x_{in,k}(t) = \sum_k x_{in,k}(t),$$

then the reaction of the output electric circuit $x_{out}(t)$ equals to the sum of reactions on each of the simple actions$x_{out,k}(t)$ separately

$$x_{in,k}(t) = \sum_k x_{in,k}(t) \tag{4.1}$$

where $x_{out}(t)$ is electric circuit reaction on the simple action $x_{out,k}(t)$ .

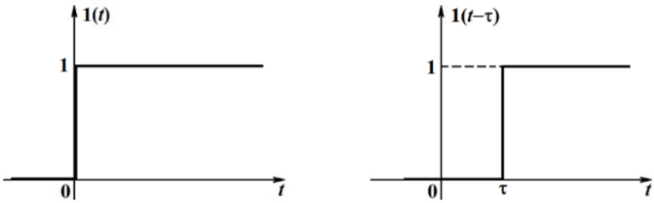
It is convenient to present the complex action as the sum of such simple actions, which reactions definition does not require much effort. Such actions are called typical.

## 4.2. Typical impulse actions

In practice, for the analysis of electric circuits the two types of typical actions are used widely: single step function and delta function.

*The single step function* (switching function, Heaviside function) 1($t$) is determined by the such relation (Fig. 4.1,$a$):

$$1(t) = \begin{cases} 0 \text{ at } t < 0; \\ 1 \text{ at } t > 0. \end{cases}$$




a b

Fig. 4.1

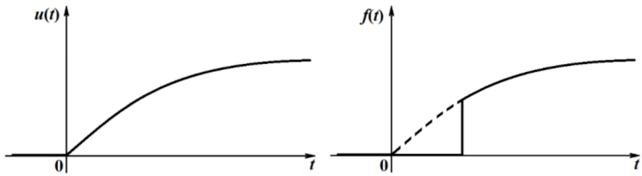
For the time value $t = 0$ function is not defined.

At the functions is shifted to the right along the time axis on the distance τ function is determined by the following relation (Fig. 4.1.b):

$$1(t-\tau)=\begin{cases}0 \text{ at } t<\tau;\\ 1 \text{ at } t>\tau.\end{cases}$$

By the function $1(t)$ using it is possible to present the different signals. For example, switching the voltage $u(t)$ (Fig. 4.2.a) at the time moment τ is expressed by expression $f(t)=u(t)1(t-\tau)$.

This process is shown in Fig.4.2.b.




a b

Fig.4.2

The rectangular impulse (Fig. 4.3, c) can be written as:

$$f(t)=1(t-t_1)-1(t-t_2). \tag{4.2}$$

Functions $1(t-t_1)$ and $1(t-t_2)$ in expression (4.2) are shown in Fig. 4.3, a and Fig. 4.3, b respectively.

The complex function of an arbitrary form can be represented approximately through the single step functions. Graphics of function *f*($t$) is shown in the Fig. 4.4. Let’s break the axis of time on small areas $\Delta t$. Then the growth of the function is

$$\Delta f_k = f(k\Delta t)-f[(k-1)\Delta t] = \\ = f(\tau)-f(\tau-\Delta t), \tag{4.3}$$

where $\tau = k\Delta t$.

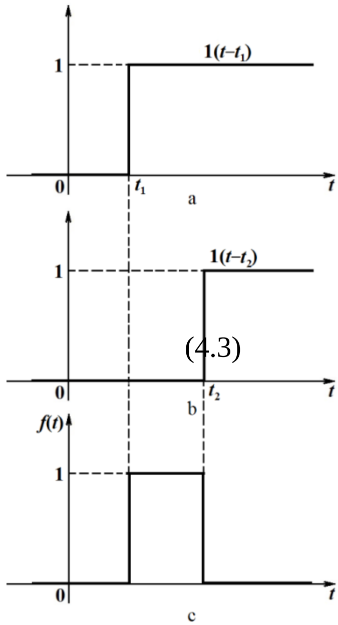



Fig. 4.3

So, expression for this function is

$$f(t) \approx f(0)1(t) + \sum_{k=1}^{n} \Delta f_k \, 1(t - k\Delta t). \qquad (4.4)$$

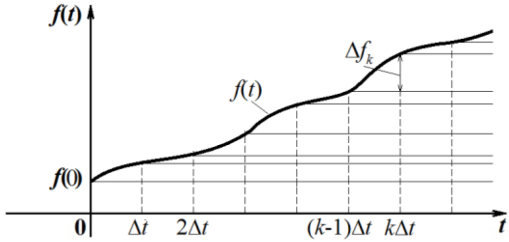



Fig. 4.4

*Delta function* (Dirac function) $\delta(t)$ is determined by the relations

$$\delta(t) = \begin{cases} 0 & at\ t \neq 0, \\ \infty & at\ t = 0, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

Delta function can be represented as an impulse with the time duration $\Delta t$ and amplitude $U_m = \frac{1}{\Delta t}$ at $\Delta t \to 0$ (Fig. 4.5), that is

$$U_m = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \to \infty \, .$$

Then the area S of the impulse time duration

$$S = U_m \Delta t = \frac{1}{\Delta t} \Delta t = 1$$

equals to unit.

At the delta function shifting on the time τ we obtain the delta function $\delta(t - \tau)$ also (Fig. 4.6).

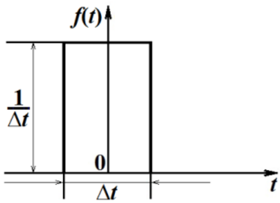



Fig. 4.5

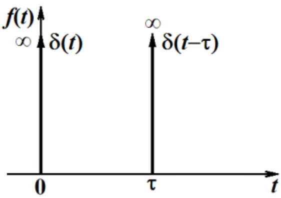



Fig. 4.6

Delta function has such properties:

$$\delta(t - \tau) = \begin{cases} 0 & at\ t \neq \tau, \\ \infty & at\ t = \tau, \end{cases} \qquad \int_{-\infty}^{\infty} \delta(t - \tau)\, dt = 1.$$

Delta function has a valuable filtering property:

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = \int_{-\Delta t}^{\Delta t} f(t)\delta(t)\,dt = \int_{-\Delta t}^{\Delta t} f(0)\delta(t)\,dt =$$

$$= f(0)\int_{-\Delta t}^{\Delta t} \delta(t)\,dt = f(0)\int_{-\infty}^{\infty} \delta(t)\,dt = f(0), \tag{4.5}$$

as far as $\lim_{\Delta t \to 0} f(t) = f(0)$.

Analogically for delta-function $\delta(t-\tau)$ property we have

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt = f(\tau). \tag{4.6}$$

The filtering property of the delta-function is shown graphically in Fig.4.7.

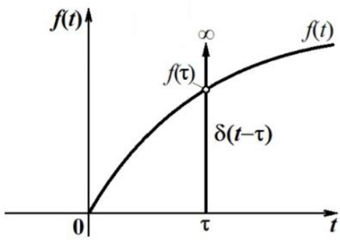



Fig.4.7

From the comparison of the single step function (see. Fig. 4.1) and delta-function (see. Fig. 4.6) it is clear that

$$\int_{-\infty}^{t} \delta(t)\,dt = 1(t),$$

$$\delta(t) = \frac{d}{dt}[1(t)] = 1'(t).$$

### 4.3 Circuit time characteristic

The time characteristic of a circuit is called a function of time, which value is determined by the reaction of the circuit on the given typical action. Such reaction depends only on the circuit's scheme, that is, it can serve as a characteristic of circuit.

The time characteristics are defined for linear circuits, which are don't have independent sources of energy at the zero initial conditions. The time characteristics are devided into two groups: transient and impulsed.

The *transient* characteristic or the transient function is determined by the reaction of the circuit on the action of the unit step function. It has the following varieties:

a) at the action of a single jump of voltage:

– if the reaction is voltage, then the characteristic is called as transient coefficient under the voltage $K_u(t)$ (dimensionless value);

– if the reaction is electric current, then the characteristic is called as transient conductivity $Y(t)$ (the unit of conductivity measurement is Siemens (Sm));

b) at the single jump of electric current:

– if the reaction is voltage, then the characteristic is called as transient resistance $Z(t)$ (the unit of the resistance measurement is Ohm);

– if the reaction is electric current, then the characteristic is called the transient transmission coefficient under electric current $K_i(t)$ (dimensionless value).

In general, the transient characteristic is denoted by $h(t)$. For individual jumps of electric parameters a jump of constant voltage from zero to 1V or jump of a direct current from zero to 1A is used.

*Impulse* characteristic or impulsed transient function is determined by the circuit reaction on the action of the delta-function form.

At the impulse characteristics calculation at the input of a electric circuit the impulses of infinite amplitude value, zero time duration and unit area is applied. So, we have the following types of impulse characteristics:

a) at the action of impulse with area in 1Vs:

– if the reaction is voltage, then the characteristic is called as impulse transient coefficient of voltage. (The unit of measurement of the impulse voltage coefficient is the unit per second (1/s));

– if the reaction is an electric current, then the characteristic is

– called as impulse conductivity. (The unit of measurement for impulse conductivity is Sm per second (Sm/s));

b) at the electric current impulse action with an area of 1As:

– if the reaction is voltage, then its characteristic is called as impulse resistance. (The unit of impulse resistance is Ohm per second (Ohm/s);

– if the reaction is a electric current, its characteristic is called as impulse transient current coefficient. (The unit of the impulse current transient coefficient is the unit per second (1/s)).

In the general case, the impulse characteristic is denoted as $a(t)$.

Let's define the relation between the transient and the impulse characteristics.

Let's firstly consider the reaction of electric circuit on the impulse action of short time duration $t_i = \Delta t$ (Fig. 4.8,$c$)

$$f_{in}(t) = U_m[1(t) - 1(t - \Delta t)], \tag{4.7}$$

here $U_m$ is impulse amplitude.

By the reaction definition of an electric circuit on a single step function $1(t)$ (Fig. 4.8,a) or $1(t - \Delta t)$ (Fig. 4.8,b) is transient characteristic $h(t)$ or $h(t - \Delta t)$.

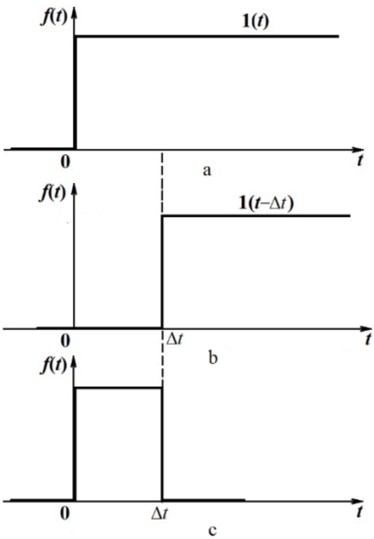



Fig. 4.8

Then by the overlay principle the reaction of an electric circuit on external action is defined by the formula (4.7)

$$f_{out}(t) = U_m[h(t) - h(t - \Delta t)] =$$
$$= \frac{h(t)-h(t-\Delta t)}{\Delta t} U_m \Delta t = \frac{\Delta h(t)}{\Delta t} S_i, \tag{4.8}$$

where $\Delta h(t) = h(t) - h(t - \Delta t)$ is growth of function $h(t)$; $S_i = U_m \Delta t$ is area of impulse.

Let's $\Delta t \to 0$ and $U_m \to \frac{1}{\Delta t}$. Then the input action goes to the delta-function according to formula (4.7), because $U_m \to \infty$ and area $S_i \to 1$. Reaction of electric circuit according to expression (4.8) is

$$f_{out}(t) = \lim_{\Delta t \to 0} \frac{\Delta h(t)}{\Delta t} S_i = h'(t).$$

By definition, the reaction of an electric circuit to a delta-function form action is a impulse characteristic, i.e.

$$a(t) = h'(t). \tag{4.9}$$

The reaction of an electric circuit to the action of small but finite time duration with the impulse area $S_i$ is determined by the expression

$$f_{out}(t) \approx a(t)S_i. \tag{4.10}$$

The approximate equality (4.10) is more accurate, then the time duration of the impulse $\Delta t$ is smaller. As it is noted earlier, the time characteristics are determined for electric circuits with zero initial conditions. Therefore, the transient characteristic $h(t)$ must be recorded as: $h(t)1(t)$.

Then, according to the expression (4.9) we have

$$a(t) = \frac{d}{dt}[h(t)1(t)] = h'^{(t)}1(t) + h(t)1'^{(t)}$$

$$= h'^{(t)}1(t) + h(0)\delta(t). \tag{4.11}$$

Expression (4.11) is called the generalized derivative. If at $t = 0$, $h(0) = 0$, then the generalized derivative coincides with the ordinary derivative (4.9).

$$h(t) = \int_{-\infty}^{t} a(t)dt.$$

Let’s consider the examples of time characteristics determination.

**Example 4.1.**

Define time characteristics for the circuit on fig.3.7.

*Solution.*

1. Transient admittance. In the input circuit is applied single step function

$$u(t) = 1(t) \tag{4.12}$$

In operation form (table 2.1)

$$1(t) = \frac{1}{p} = U(p).$$

Expression of operator electric current in electric circuit is

$$i(t) \doteqdot I(p) = \frac{U(p)}{Z(p)} = \frac{1}{p}\frac{1}{r+\frac{1}{pC}} = \frac{1}{r}\frac{1}{p+\frac{1}{rC}}. \qquad (4.13)$$

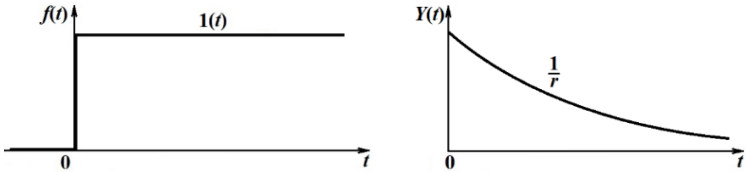
The original form of electric current in the electric circuit according to the Table 2.1 is expressed in the form

$$i(t) = \frac{1}{r}e^{-\frac{1}{rC}}.$$

Now the transient conductivity has the form

$$Y(t) = \frac{i(t)}{u(t)} = \frac{i(t)}{1} = \frac{1}{r}e^{-\frac{1}{rC}}. \qquad (4.14)$$

The graph of transient conductivity is depicted in Fig. 4.9, b and the input action is described with a single step function (Fig. 4.9, a).




a b

Fig. 4.9

2. The impulse conductivity. At the input part of the electric circuit a voltage impulse in the delta-function form is acted

$$u(t) = \delta(t) \qquad (4.15)$$

In the operator form according to Tab. 2.1 we have

$$\delta(t) \doteqdot 1 = U(p).$$

Expression of the current operator in the electric circuit is

$$L[i(t)] = I(p) = \frac{U(p)}{Z(p)} = \frac{1}{p}\frac{1}{r+\frac{1}{pC}} = \frac{1}{r}\frac{1}{p+\frac{1}{rC}} =$$

$$= \frac{1}{r}\frac{p+\frac{1}{rC}-\frac{1}{rC}}{p+\frac{1}{rC}} = \frac{1}{r}\left(1-\frac{1}{rC}\frac{1}{p+\frac{1}{rC}}\right). \qquad (4.16)$$

Here is the degree of numerator and denominator in the image forms were the same. Therefore, the allocated whole part in the expression is the unit, and the degree of the numerator was lower, and the expression for operator image became the correct fraction.

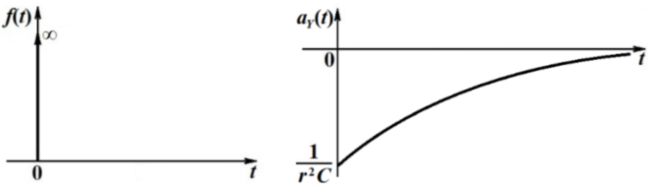
Expression for the operator image of impulse conductivity is

$$a_Y(p) = \frac{I(p)}{U(p)} = \frac{1}{r}\left(1 - \frac{1}{rC}\frac{1}{p + \frac{1}{rC}}\right).$$

Expression for the original of impulse conductivity is

$$a_Y(p) = a_Y(t) = \frac{1}{r}\left[\delta(t) - \frac{1}{rC}e^{\frac{1}{rC}}\right]. \qquad (4.17)$$

The graph of impulse conductivity function is depicted in Fig. 4.10, b; the form of the input action is delta-function $\delta(t)$ is depicted in Fig. 4.10, a.




a b

Fig. 4.10

The impulse characteristic can be found by a generalized derivative of the transient characteristic (4.11). By using the expression (4.14) we obtain:

$$Y'(t) = -\frac{1}{r^2C}e^{-\frac{1}{rC}}; \qquad Y(0) = \frac{1}{r}.$$

So, from fornula (4.11) we have

$$a_Y(t) = -\frac{1}{r^2C}e^{-\frac{1}{rC}}1(t) + \frac{1}{r}\delta(t) = \frac{1}{r}\left[\delta(t) - \frac{1}{rC}e^{-\frac{1}{rC}}\right],$$

which coincides with the expression (4.17).

**Example 4.2.**

To define the time characteristics for an electric circuit in Fig. 4.11.

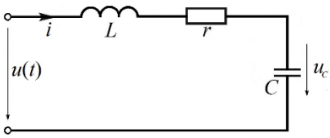



Fig. 4.11

*Solution.*

1. A single step function which is described with expression (4.12) acts in the input part of the electric circuit. The current in the electric circuit in oscillatory mode is determined similarly to the expression (2.46).

$$i(t) = \frac{1}{\omega_{fre} L} e^{-\delta t} \sin \omega_{fre} t,$$

expression for the voltage on the condenser $u_C(t)$ is similar to the expression (2.54)

$$u_C(t) = 1 - \frac{\omega_0}{\omega_{fre}} e^{-\delta t} \sin\left(\omega_{fre} t + \alpha\right).$$

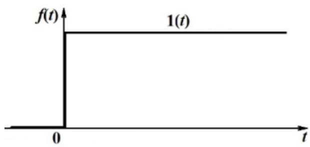
So, expression for the transient conductivity $Y(t)$ is

$$Y(t) = \frac{i(t)}{u(t)} = \frac{1}{\omega_{fre} L} e^{-\delta t} \sin \omega_{fre} t. \qquad (4.18)$$

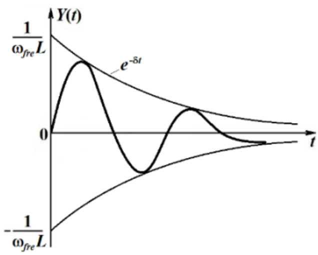
Expression for the transient voltage coefficient is

$$K_U(t) = \frac{u_C(t)}{u(t)} = 1 - \frac{\omega_0}{\omega_{fre}} e^{-\delta t} \sin\left(\omega_{fre} t + \alpha\right). \qquad (4.19)$$

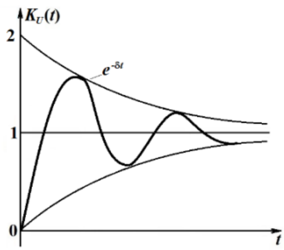
Graphs of transient characteristics $Y(t)$ and $K_U(t)$ are shown in Fig. 4.12, b and Fig. 4.12, c (the input action is described with the unit step function $1(t)$ (Fig.4.12,a).




a




b




c

Fig.4.12

2. Imulse characteristics. At the input part of the electric circuit acts a voltage impulse in the delta function form in accordance with the expression (4.15). The impulse conductivity is defined as a generalized derivative of transient conductivity. From the expression (4.18) we have

$$Y'(t) = \frac{1}{\omega_{fre}L}\left[e^{-\delta t}(-\delta)\sin\omega_{fre}t + \omega_{fre}e^{-\delta t}\cos\omega_{fre}t\right] =$$

$$= \frac{\omega_0}{\omega_{fre}L}e^{-\delta t}\cos\left(\omega_{fre}t + \frac{\pi}{2} - \alpha\right); \qquad Y(0) = 0.$$

Then according to the formula (4.11) we have

$$a_Y(t) = \frac{\omega_0}{\omega_{fre}L}e^{-\delta t}\cos\left(\omega_{fre}t + \frac{\pi}{2} - \alpha\right).$$

Voltage impulse transient coefficient is determined as a generalized derivative of the voltage transfer coefficient also. From expression (4.19) we have

$$K'_U(t) = \frac{\omega_0}{\omega_{fre}}\left[e^{-\delta t}(-\delta)\sin(\omega_{fre}t + \alpha) +\right.$$

$$\left.+\omega_{fre}e^{-\delta t}\cos(\omega_{fre}t + \alpha)\right] = \frac{\omega_0}{\omega_{fre}}e^{-\delta t}\sin\omega_{fre}t;$$

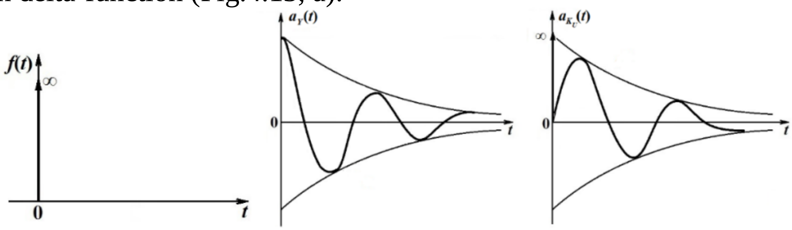
$$K_U(0) = 1 - \frac{\omega_0}{\omega_{fre}}\sin\alpha.$$

Then by the formula (4.11) we have

$$a_{K_U}(t) = 1(t)\frac{\omega_0}{\omega_{fre}}e^{-\delta t}\sin\omega_{fre}t + \left(1 - \frac{\omega_0}{\omega_{fre}}\sin\alpha\right)\delta(t) =$$

$$= \frac{\omega_0}{\omega_{fre}}e^{-\delta t}\sin\omega_{fre}t + \left(1 - \frac{\omega_0}{\omega_{fre}}\sin\alpha\right)\delta(t).$$

Graphs of impulse characteristics $a_Y(t)$ and $a_{K_U}(t)$ are shown in Fig. 4.13, b, Fig. 4.13, c. The input action is mathematically described with delta-function (Fig.4.13, a).




a b c

Fig.4.13

**Problem 4.1.**

Calculate and analyze time characteristics $h_{Y_{21}}(t)$, $a_{Y_{21}}(t)$ for electrical circuit of the second order (fig. P.3.2).

*Solution.*

Let's find transient and pulse characteristics $h_{Y_{21}}(t)$, $a_{Y_{21}}(t)$ using (P.3.1.)

Transient characteristic:

$$h_{Y_{21}}(p) = \frac{1}{p}Y_{21}(p) = \frac{1}{(r_1 + r_2)r_3C(p^2 + 2\delta p + \omega_0^2)}.$$

Its original

$$h_{Y_{21}}(p) = \frac{1}{(r_1 + r_2)r_3C(p_1 - p_2)}(e^{p_1 \mathrm{t}} - e^{p_2 \mathrm{t}}). \qquad \text{(P.4.1)}$$

Accounting numerical calculation, we get from (P.4.1)

$$h_{Y_{21}}(p) = \frac{1}{(10 + 10) \cdot 4 \cdot 10 \cdot 10^{-6}(-3{,}82 \cdot 10^3 + 26{,}18 \cdot 10^3)} \times \qquad \text{(P.4.2)}$$
$$\times \left(e^{-3{,}82 \cdot 10^3 t} - e^{-26{,}18 \cdot 10^3 t}\right) = 0{,}56\left(e^{-\frac{t}{0{,}262 \cdot 10^{-3}}} - e^{-\frac{t}{0{,}038 \cdot 10^{-3}}}\right).$$

Impulse characteristic.

$$a_{Y_{21}}(t) = 1 \cdot Y_{21}(p) = \frac{1}{(r_1 + r_2)r_3C} \cdot \frac{p}{p^2 + 2\delta p + \omega_0^2}. \qquad \text{(P.4.3)}$$

Let's find original from (P.4.3). As

$$a_{Y_{21}}(p) = \frac{1}{(r_1 + r_2)r_3C} \cdot \frac{p}{(p - p_1)(p - p_2)},$$

Then

$$a_{Y_{21}}(p) = \frac{1}{(r_1 + r_2)r_3C}\left(\frac{p_1}{p_1 - p_2}e^{p_1 t} + \frac{p_2}{p_2 - p_1}e^{p_2 t}\right) =$$
$$= \frac{1}{(r_1 + r_2)r_3C(p_1 - p_2)}(p_1 e^{p_1 \mathrm{t}} - p_2 e^{p_2 \mathrm{t}}). \qquad \text{(P.4.4)}$$

Accounting calculation for i1(t), we get from (P.4.4)

$$a_{Y_{21}}(t) = \frac{1}{(10 + 10) \cdot 4 \cdot 10 \cdot 10^{-6}(-3{,}82 \cdot 10^3 + 26{,}18 \cdot 10^3)} \times$$
$$\times \left(-3{,}82 \cdot 10^3 e^{-3{,}82 \cdot 10^3 t} + 26{,}18 \cdot 10^3 e^{-26{,}18 \cdot 10^3 t}\right) =$$
$$= \left(1466 e^{-\frac{t}{0{,}62 \cdot 10^{-3}}} - 214 e^{-\frac{t}{0{,}262 \cdot 10^{-3}}}\right) \text{ Cm/c.} \qquad \text{(P.4.5)}$$

Graphic of transient characteristic is shown in fig. P.4.1 of pulse characteristic – in fig. P.4.2

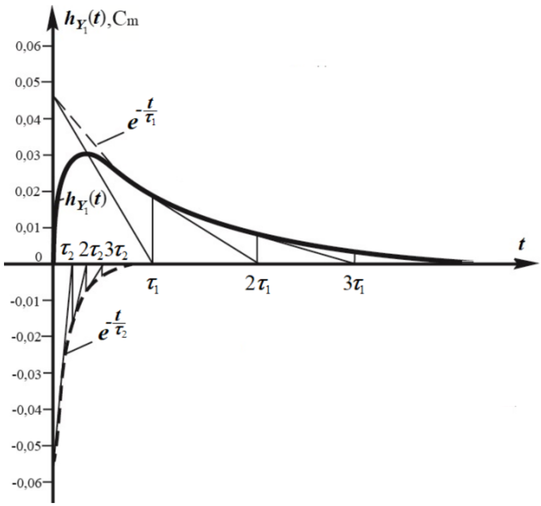



Fig. P.4.1

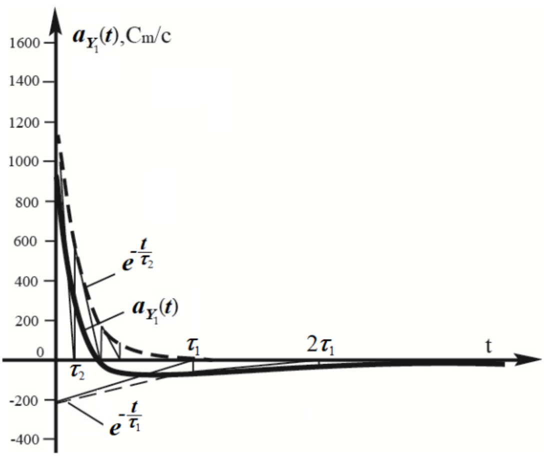



Fig. P.4.2

### 4.4. The convolution integral

In Section 4.2, it was shown how the function $f(t)$ of the arbitrary form can be approximated by the sum of the shifting among themselves stepped functions with different amplitudes (see Figure 4.4). Such a function may also be represented by the sum of the rectangular impulses of short duration $\Delta t$ in time between them (Fig. 4.14):

$$f_{in}(t) \approx \sum_{k=0}^{n} f_{in,k}(t)\,, \tag{4.20}$$

where

$$f_{in,k}(t) = f_{in}(k\Delta t)[1(t - k\Delta t) - 1(t - (k+1)\Delta t)] =$$
$$= f_{in}(\tau)[1(t-\tau) - 1(t - \tau - \Delta\tau)].$$

Here, as in the expression (4.3), it is accepted that $\tau = k\Delta t$ and $\Delta t = \Delta\tau$ which is true for small time intervals $\Delta t$.

In the expression (4.20) $f_{in,k}(t)$ is the impulse time duration $\Delta t$ (shaded time interval in Fig. 4.14) with amplitude $f_{in}(k\Delta t)$. That is, the function $f_{in}(t)$ can be approximated by the sum of such impulses $f_{in}(k\Delta t)$. Thus, the initial action can be expressed with the sum of impulses of short duration.

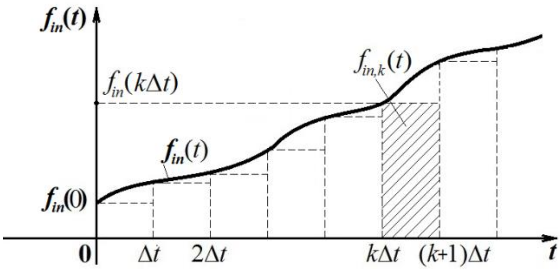



Fig. 4.14

To determine the electric circuit reaction the expression (4.10) and the electric circuit reaction into the impulse action $f_{in,k}(t)$ are used

$$f_{out,k}(t) \approx a(t - k\Delta t)S_{ik}\,,$$

where $S_{ik}$ is area of k-th $f_{in}(t)$. Then

$$f_{out,k}(t) \approx a(t-\tau)f_{in}(\tau)\Delta\tau.$$

The total reaction on action $f_{out,k}(t)$ is determined according to the superposition principle (4.1)

$$f_{out}(t) \approx \sum_{k=0}^{n} f_{out,k}(t) = \sum_{\tau=k\Delta t=0}^{\tau=n\Delta t} f_{in}(\tau)a(t-\tau)\Delta\tau.$$

Here sum on k is replaced on sum on time from $\tau = k\Delta t = 0$ at $k = 0$ to $\tau = n\Delta t$ at $k = n$.

At time interval $\Delta t$ tends to zero ($\Delta t \to 0$) we have expression:

$$f_{out}(t) = \int_{0}^{t} f_{in}(\tau)a(t-\tau)d\tau\,. \qquad (4.21)$$

The integral (4.21) is analogous to the convolution integral. It is also called the Duhamel integral. Here the integration is executed by variable τ and t is fixed time point in which the reaction $f_{out}(t)$ is determined. Formula (4.21) shows that the reaction of the electric circuit $f_{out}(t)$ at time moment $t$ is defined as the sum of reactions of this electric circuit at this moment of time from all actions on the electric circuit acting on its input clamps at all moments of time preceding to the time moment $t$.

In the same way as formula (2.25), the second form of the Duamel integral can be written as:

$$f_{out}(t) = \int_{0}^{1} f_{in}(t-\tau)a(\tau)d\tau. \qquad (4.22)$$

The reaction of the circuit $f_{out}(t)$ on the complex action $f_{in}(t)$ can be determined by the transient characteristic $h(t)$. As already noted, the input action $f_{in}(t)$ can be represented by the sum of partial actions $f_{in,k}(t)$, which is the product of the function growth on the k-th interval and per unit step function that is late on the time interval $k\Delta t$ (see Fig. 4.4), that is by the expression (4.4)

$$f_{in}(t) = f_{in}(0)1(t) + \sum_{k=1}^{n} \Delta f_{in,k} 1(t-k\Delta t).$$

The reaction $f_{out,k}(t)$ of electric circuit on the action $f_{in,k}(t)$ is the product of the increment of the input action $\Delta f_{in,k}$ on the transient characteristic $h(t-k\Delta t)$, which is late on the interval $k(\Delta t)$.

$$\Delta f_{out,k}(t) = \Delta f_{in,k} h(t-k\Delta t).$$

It can be written down with the second order of smallness as

$$\Delta f_{in,k} \approx f'_{in}(k\Delta t)\Delta t. \qquad (4.23)$$

Here by summation of partial reactions with take into account expression (4.23) we have

$$f_{out}(t) \approx \sum_{k=0}^{n} \Delta f_{out,k}(t) = \tag{4.24}$$

$$= f_{in}(0)h(t) + \sum_{k=1}^{n} f'_{in}(k\Delta t)\Delta t\, h(t - k\Delta t).$$

If now to increase the number of $n$ intervals, correspondingly reducing the duration of the time interval $\Delta t$, then the input action approaches the smooth curve $f_{in}(t)$ and the sum in the expression (4.24) pass to the integral, and the approximate equality equals the exact:

$$f_{out}(t) = f_{in}(0)h(t) + \int_0^t f'_{in}(\tau)\mathrm{h}(\mathrm{t} - \tau)dt\,, \tag{4.25}$$

$k\Delta t = \tau\,;\qquad \Delta t \to dt.$

This is the third form of the Duamel integral.

By using the identity of the integrals in the convolution formulas (2.25) and (2.26), one can write the fourth form of the Duamel integral

$$f_{out}(t) = f_{in}(0)h(t) + \int_0^t f'_{in}(t - \tau)h(\tau)dt$$

Let's apply to the integral in the right side of formula (4.25) the integration rule by parts (2.5), where

$u = h(t - \tau);\ dV = f'_{in}(\tau)d\tau;\ \ V = f_{in}(\tau);\ \ du = -h'(t - \tau)d\tau.$

Then

$$\int_0^t f'_{in}(\tau)h(t - \tau)\partial\tau = h(t - \tau)f_{in}(\tau)\Big|_0^t - \int_0^t f_{in}(\tau)[-h'(t - \tau)\partial\tau] =$$

$$= f_{in}(t)h(0) - f_{in}(0)h(t) + \int_0^t f_{in}(\tau)h'(t - \tau)dt. \tag{4.26}$$

By substituting the expression (4.26) into expression (4.25), we get

$$f_{out}(t) = f_{in}(t)h(0) + \int_0^t f_{in}(\tau)h'(t - \tau)d\tau\,. \tag{4.27}$$

This is the fifth form of the Duamel integral.

By using the identity of the integrals (3.25) and (3.26) again, we have from expression (4.27):

$$f_{out}(t) = f_{in}(t)h(0) + \int_0^t f_{in}(t-\tau)h'^{(\tau)}d\tau.$$

This is sixth form Duhamel integral.

The Duamel integrals are used in calculating the reaction of circuit to the action of an arbitrary form in the following sequence:

1) To calculate the appropriate time characteristic;
2) To determine the required reaction of electric circuit on action by the formulas of the Duamel integral.

**Example 4.3.**

The voltage of arbitrary form is applied to the input of electric circuit (Fig. 3.7)

$$u(t) = U_m e^{-\alpha t}. \qquad (4.28)$$

Let's find the voltage $U_c$ ($t$) on the capacitor $C$.

*Solution.*

1. If the input action and the reaction of the electric circuit is a voltage, then the time characteristic of the circuit is found in the form of a voltage transient coefficient. To determine the reaction of the circuit we use the formula (4.22), where $f_{out}(t) = u_c(t)$, $f_{out}(t) = u(t)$, $f_{in}(t - \tau) = u(t - \tau)$.

The impulse characteristic $a(t)$ is the impulse voltage transient coefficient $a_{K_U}(t)$; that is, the expression (4.22) has the form:

$$u_c(t) = \int_0^t u(t-\tau)a_{K_U}(\tau)d\tau. \qquad (4.29)$$

For determination the value of $a_{K_U}(t)$ to write expression for $a_{K_U}(p)$, by using the value of $I(p)$ obtained in Example 4.1 according to formula (4.16)

$$a_{K_U}(p) = I(p)\frac{1}{pC} = \frac{1}{r}\frac{p}{p+\frac{1}{rC}}\frac{1}{pC} = \frac{1}{rC}\frac{1}{p+\frac{1}{rC}}. \qquad (4.30)$$

The original of the expression (4.30) according to Table 2.1 looks like

$$a_{K_U}(t) = \frac{1}{rC} e^{-\frac{1}{rC}}. \qquad (4.31)$$

2. For the integral (4.29) we obtain from the expressions (4.28) and (4.31)

$$u(t-\tau) = U_m e^{-\alpha(t-\tau)}; \quad a_{ku}(\tau) = \frac{1}{rC} e^{-\frac{\tau}{rC}}.$$

Then the voltage of the capacitor $C$ is

$$u_c(t) = \int_0^t U_m e^{-\alpha(t-\tau)} \frac{1}{rC} e^{-\frac{\tau}{rC}} d\tau = \frac{U_m}{rC} e^{-\alpha t} \int_0^t e^{\left(\alpha - \frac{1}{rC}\right)\tau} dt =$$

$$= \frac{U_m}{rC} e^{-\alpha t} \frac{rC}{\alpha rC - 1} e^{\left(\alpha - \frac{1}{rC}\right)\tau} \Big|_0^t = \frac{U_m}{rC} e^{-\alpha t\left[e^{\left(\alpha - \frac{1}{rC}\right)t} - 1\right]} =$$

$$= \frac{U_m}{1 - \alpha rC}\left(e^{-\alpha t} - e^{-\frac{1}{rC}}\right). \qquad (4.32)$$

To determine the voltage $u_c(t)$ by using the integral expression (4.25), where $f_{in}(0) = u(0)$. The transient characteristic $h(t)$ is the transient coefficient of transition over the voltage $K_U(t)$. So, $h(t-\tau) = K_U(t-\tau)$; that is, the expression (4.25) takes the form:

$$u_c(t) = u(0)K_U(t) + \int_0^t \frac{d}{d\tau} u(\tau) K_U(t-\tau) d\tau. \qquad (4.33)$$

To solve the problem, one must determine the transient characteristic $K_U(t)$. To do this, by using the result of Example 4.1 for $I(p)$ value by formula (4.13) we write the expression for $K_U(p)$:

$$K_U(p) = I(p)\frac{1}{pC} = \frac{1}{r}\frac{1}{p + \frac{1}{rC}}\frac{1}{pC} = \frac{1}{rC}\frac{1}{p(p + \frac{1}{rC})}. \qquad (4.34)$$

The original of the expression (4.34) according to Table 2.1 looks like

$$K_U(t) = 1 - e^{-\frac{1}{rC}}. \qquad (4.35)$$

According to the formulas (4.25), (4.28) and (4.35) we obtain expressions

$$u(0) = U_m; \quad u_{in}(\tau) = U_m e^{-\alpha\tau};$$

$$\frac{d}{d\tau}[u_{in}(\tau)] = U_m e^{-\alpha\tau}(-\alpha) = -\alpha U_m e^{-\alpha\tau}; \qquad (4.36)$$

$$K_u(t-\tau) = 1 - e^{-\frac{1}{rC}(t-\tau)}.$$

Then the voltage of the capacitor $C$ according to formula (4.33) takes the form

$$u_c(t) = U_m\left(1 - e^{-\frac{1}{rC}}\right) + \int_0^t (-\alpha U_m e^{-\alpha\tau})\left[1 - e^{-\frac{1}{rC}(t-\tau)}\right] d\tau =$$

$$= U_m\left(1 - e^{-\frac{1}{rC}}\right) - \alpha U_m \int_0^t e^{-\alpha\tau}\left[1 - e^{-\frac{1}{rC}(t-\tau)}\right] d\tau =$$

$$= U_m\left(1 - e^{-\frac{1}{rC}}\right) - \alpha U_m \int_0^t e^{-\alpha\tau}\, d\tau + \alpha U_m e^{-\frac{1}{rC}} \int_0^t e^{(\frac{1}{rC}-\alpha)\tau}\, d\tau =$$

$$= U_m\left(e^{-\alpha\tau} - e^{-\frac{1}{rC}}\right) + \frac{\alpha rC}{1 - \alpha rC} U_m\left(e^{-\alpha\tau} - e^{-\frac{1}{rC}}\right) = \tag{4.37}$$

$$= \frac{U_m}{1-\alpha rC}\left(e^{-\alpha\tau} - e^{-\frac{1}{rC}}\right).$$

The result (4.37) coincides with expression (4.32). By comparing the formula (4.37) and expression for electric current (3.7) obtained in Example 3.2, we can see that the diagram $u_c(t)$ for the electric circuit expressed in Fig. 3.7 looks like in Fig. 3.4.

**Example 4.4.**

At the electric circuit input (Fig. 3.7) a voltage impulse is applied the form of which is shown in Fig. 3.8, where in the time interval [0-$t_1$] the input voltage varies exponentially

$$u(t) = U_0 e^{-\alpha t}. \tag{4.38}$$

To determine the electric current $i(t)$ in circuit.

*Solution* of problem we find by the formula of Duamel integral (4.25).

As transient characteristic, obviously, will be transient conductivity $Y(t)$ (4.14)). Then expression (4.25) takes the form

$$i(t) = u(0)Y(t) + \int_0^t u'(\tau)Y(t-\tau)d\tau\,. \tag{4.39}$$

Let's define the components of the expression (4.39). Similarly, to expression (4.36), we have

$$u(0) = U_0. \tag{4.40}$$

From formula (4.38) we write

$$u(\tau) = U_0 e^{-\alpha\tau}; \tag{4.41}$$

$$u'(\tau) = -\alpha U_0 e^{-\alpha\tau}. \tag{4.42}$$

From relation (4.14) we have

$$Y(t-\tau) = \frac{1}{r} e^{\frac{t-\tau}{rC}}. \tag{4.43}$$

Expression for voltage $u(t)$ (Fig.3.8) is:

$$u(t) = \begin{cases} 0, & \text{at } t < 0 \\ U_o e^{-\alpha\tau}, & \text{at } 0 \le t \le t_1 \\ 0, & \text{at } t \ge t_1. \end{cases} \tag{4.44}$$

Expressions for the electric current $i(t)$ are defined in each interval separately.

1. In the time interval $t < 0$ the input action $u(t) = 0$, so electric current reaction $i(t)$ equals to zero also.
2. In the time interval $0 \le t \le t_1$ according to expressions (4.14), (4.39), (4.40), (4.42) and (4.43) we get:

$$\begin{aligned} i(t) &= U_0 \frac{1}{r} e^{-\frac{t}{rC}} - \int_0^t \alpha U_0 e^{-\alpha\tau} \frac{1}{r} e^{\frac{t-\tau}{rC}} dt = \\ &= \frac{U_0}{r} e^{-\frac{t}{rC}} - \frac{\alpha U_0}{r} e^{-\frac{t}{rC}} \int_0^t e^{\left(\frac{1}{rC}-\alpha\right)\tau} dt = \\ &= \frac{U_0}{r} e^{-\frac{t}{rC}} - \frac{U_0}{r} \frac{\alpha rC}{1-\alpha rC}\left(e^{-\alpha\tau} - e^{-\frac{t}{rC}}\right) = \\ &= \frac{U_0}{r(1-\alpha rC)}\left(e^{-\frac{t}{rC}} - \alpha rC e^{-\alpha \mathrm{t}}\right). \end{aligned} \tag{4.45}$$

3. In the time interval $t \ge t_1$ the expression for electric current $i(t)$ is getting by subtracting from expression (4.45) at the time moment $t = t_1$ the expression for electric circuit reaction on negative jump of input voltage:

$$u(t_1) = U_0 e^{-\alpha t}. \tag{4.46}$$

This reaction according to expressions (4.14) and (4.46) has the form:

$$i(t) = u(t_1)Y(t - t_1) = U_0 e^{-\alpha t_1} \frac{1}{r} e^{\frac{(t-t_1)}{rC}}$$
$$= \frac{U_0}{r} e^{(\frac{1}{rC} - \alpha)t_1} e^{-\frac{t}{rC}}. \qquad (4.47)$$

And expression for electric current $i(t)$ in the time interval $t > t_1$ is determined from equations (4.14), (4.39), (4.40), (4.42) and (4.43)

$$i(t) = u_0 \frac{1}{r} e^{-\frac{t}{rC}} - \int_0^{t_1} \alpha U_0 \, e^{-\alpha\tau} \frac{1}{r} e^{-\frac{t-\tau}{rC}} d\tau - \frac{U_0}{r} e^{\left(\frac{1}{rC} - \alpha\right)t_1} e^{-\frac{t}{rC}} =$$
$$(4.48)$$
$$= \frac{U_0}{r} e^{-\frac{t}{rC}} - \frac{\alpha U_0}{r} e^{-\frac{t}{rC}} \int_0^{t_1} e^{\left(\frac{1}{rC} - \alpha\right)\tau} dt - \frac{U_0}{r} e^{\left(\frac{1}{rC} - \alpha\right)t_1} e^{-\frac{t}{rC}} =$$
$$= \frac{u_0}{r(1 - \alpha rC)} \left[1 - e^{(\frac{1}{rC} - \alpha)t_1}\right] e^{-\frac{t}{rC}}.$$

Expressions (4.45) and (4.48) coincide with the corresponding expressions (3.15) and (3.16).

Let's analyze the electric current $i(t)$ change over the time in each time intervals.

In the time interval $t < 0$, as already was noted $i(t) = 0$.

At the time $t = 0$, according to the expression (4.45), the electric current jumps up to the value of $i(0) = \frac{U_0}{r}$.

In the time interval $0 \leq \tau < t_1$, the electric current $i(t)$ varies according to the formulas (4.45). Here are the following options:

1. $\alpha < \frac{1}{rC}$. Then $\alpha rC < 1$ and exponential function $e^{-\frac{t}{rC}}$ attenuates faster than function $e^{-\alpha\tau}$. However, the maximum value of the first exponent is $(\frac{u_0}{r(1-\alpha rC)})$ at $t = 0$ and it is more than the maximum value of the second exponent $(\frac{U_0 \alpha rC}{r(1-\alpha rC)})$. Their difference is equals $\frac{U_0}{r}$. Therefore, the electric current $i(t)$ falls from $\frac{U_0}{r}$ при $t = 0$ to its value according to expression (4.45) at $t = t_1$:

$i(t_1^-) = \frac{U_0}{r(1-\alpha rC)}(e^{-\frac{t_1}{rC}} - \alpha rC e^{-\alpha t_1})$.

And at $t = t_1$:

$$i(t_1^+) = \frac{U_0}{r(1-\alpha rC)}(e^{-\frac{t_1}{rC}} - e^{-\alpha t_1}) \ .$$

Obviously, this value is negative and the value of the electric current jump is

$$i(t_1^+) - i(t_1^-) = -\frac{U_0}{r}e^{-\alpha t_1}. \qquad (4.49)$$

2. Let's $\alpha = \frac{1}{rC}$. Then $\alpha rC = 1$ and exponential functions $e^{-\frac{t}{rC}}$ and $e^{-\alpha t}$ attenuate with equal velocity. According to the equation (4.45) the expression for electric current is

$$i(t) = \frac{U_0}{r}e^{-\alpha t} = \frac{U_0}{r}e^{-\frac{t}{rC}}.$$

At the time moment $t_1$ the electric current value falls to the value

$$i(t_1^-) = \frac{U_0}{r}e^{-\alpha t_1} = \frac{U_0}{r}e^{-\frac{t_1}{rC}}.$$

This value is positive.

At time moment $t = t_1$ according to the expression (4.49) electric current change to zero by jump on the value

$$-\frac{U_0}{r}e^{-\frac{t}{rC}} = \frac{U_0}{r}e^{-\alpha t}.$$

3. At $\alpha > \frac{1}{rC}$ Then $\alpha rC > 1$ and exponential function $e^{-\alpha t}$ attenuates faster than function $e^{-\frac{t}{rC}}$. Expressions (4.45) can be rewritten as follows:

$$i(t) = \frac{U_0}{r(\alpha rC-1)}\left(\alpha rCe^{-\alpha t} - e^{-\frac{t}{rC}}\right). \qquad (4.50)$$

Here the maximum value of the first exponent $(\frac{U_0}{r(1-\alpha rC))})$ more than maximal value of the second exponent $(\frac{U_0}{r(\alpha rC-1)})$ also. Their difference equals to $\frac{U_0}{r}$. Thus electric current value $i(t)$ falls from value $\frac{U_0}{r}$ to its value according to the expression (4.50) at $t = t_1$:

$$i(t_1^-) = \frac{U_0}{r(\alpha rC - 1)}\left(\alpha rCe^{-\alpha t_1} - e^{-\frac{t_1}{rC}}\right).$$

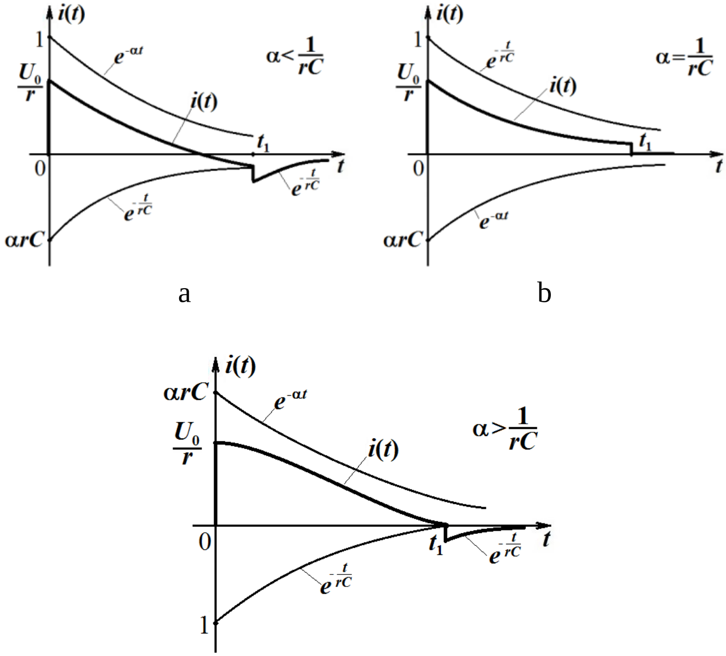
This value may become negative with a sufficiently large value $t_1$.

At $t = t_1$ the electric current falls by jump to the value according to formula (4.48):

$$i(t_1^+) = \frac{U_0}{r(\alpha rC - 1)}\left(e^{-\alpha t_1} - e^{-\frac{t_1}{rC}}\right).$$

Obviously, this value is negative. The value of the current jump is also determined by the expression (4.47).

In the time interval $t > t_1$ electric current $i(t)$ at $\alpha < \frac{1}{rC}$ or at $\alpha > \frac{1}{rC}$ falls to zero in accordance with the expression (4.48), while having a negative direction. At $\alpha = \frac{1}{rC}$ the electric current remains zero value for all $t > t_1$. Forms of electric current $i(t)$ curves at $\alpha < \frac{1}{rC}$, $\alpha > \frac{1}{rC}$, and $\alpha = \frac{1}{rC}$ are shown in Fig. 4.15,*a*,*b*,*c*.



c

Fig. 4.15

Electric current $i(t)$ in this example we can find other expressions of the Duhamel integral, for example, the expression (4.21). Here $u(\tau)$ value is determined from equation (4.41). The impulse characteristic, obviously, is an impulse conductivity $aY(t)$ from relation (4.17).

Then the expression (4.21) looks like

$$i(t) = \int_0^t u(\tau) aY(t-\tau) d\tau. \qquad (4.51)$$

Taking into account the expression (4.44), we can find the expression for electric current $i(t)$ separately for each time intervals.

1. In the time interval $t < 0$ the input action $u(t) = 0$, thus reaction i$(t)$ equals to zero also.

2. For time interval $0 \le t < t_1$ according to relations (4.17), (4.41) and (4.51) we have

$$aY(t-\tau) = \frac{1}{r}\left[\delta(t-\tau) - \frac{1}{rC} e^{-\frac{t-\tau}{rC}}\right],$$

$$i(t) = \int_0^t U_0 e^{-\alpha\tau} \frac{1}{r}\left[\delta(t-\tau) - \frac{1}{rC} e^{-\frac{t-\tau}{rC}}\right] d\tau =$$

$$\frac{U_0}{r}\left[\int_0^t e^{-\alpha\tau}\delta(t-\tau)dt - \frac{1}{rC}\int_0^t e^{-\alpha\tau}\right]. \qquad (4.52)$$

The first integral in square brackets is found according to the filtering property of the delta function

$$\int_0^t e^{-\alpha\tau}\delta(t-\tau)dt = e^{-\alpha\tau},$$

then

$$i(t) = \frac{U_0}{r}\left[e^{-\alpha\tau} - \frac{1}{rC}\int_0^t e^{(\frac{1}{rC}-\alpha)\tau} dt\right] =$$

$$= \frac{U_0}{r}\left[e^{-\alpha\tau} + \frac{1}{1-\alpha rC}\left(e^{-\frac{t}{rC}} - e^{-\alpha\tau}\right)\right] =$$

$$= \frac{U_0}{r(1-\alpha rC)}\left(e^{-\frac{t}{rC}} - \alpha rC e^{-\alpha\tau}\right).$$

4.In the time interval $t > t_1$ expression for electric current $i(t)$ we can find by replacing the upper limit at integration in expression (4.52) into $t_1$:

$$i(t) = \int_0^{t_1} U_0 e^{-\alpha\tau} \frac{1}{r}\left[\delta(t-\tau) - \frac{1}{rC}e^{-\frac{t-\tau}{rC}}\right]dt =$$

$$= \frac{U_0}{r}\left[\int_0^{t_1} e^{-\alpha\tau}\delta(t-\tau)dt - \frac{1}{rC}\int_0^{t_1} e^{-\alpha\tau}\, e^{-\frac{t-\tau}{rC}}dt\right];$$

The first integral in square brackets is calculated within the time interval $0 \div t_1$ and delta function $\delta(t-\tau)$ acts at $t > t_1$, therefore, for the filtering property of the delta functions (4.5) and (4.6) we have

$$\int_0^{t_1} e^{-\alpha\tau}\delta(t-\tau)dt = 0.$$

So,

$$i(t) = -\frac{U_0}{r}\frac{1}{rC}e^{-\frac{t}{rC}}\int_0^{t_1} e^{\left(\frac{1}{rC}-\alpha\right)\tau}dt =$$

$$= \frac{U_0}{r(1-\alpha rC)}\left[1 - e^{\left(\frac{1}{rC}-\alpha\right)t_1}\right]e^{-\frac{t}{rC}}.$$

This expression coincides with expression according to formula (4.48).

**Problem 4.3.**

Calculate and analyze (in common case) of output voltage for linear circuit of the first order, using convolution method (Duhamel integral) by impulse action of complicated form. Calculation circuit is shown in fig. P.4.8, form of input signal – in fig. P.4.9. It's necessary calculate voltage $u_L$ of inductance $L$; $u_L = u_2$.

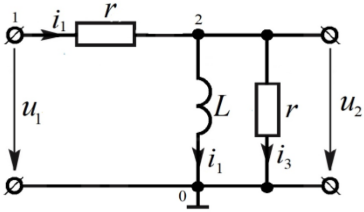



Fig.P.4.8

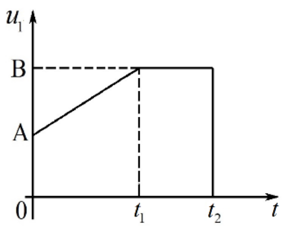



Fig.P.4.9

*Solution.*

Let's define transient characteristic $h_{K_U}(t)$ for the given circuit (Fig.P.4.8). In operational form

$$h_{K_U}(p) = K_{U_{21}}(p)\frac{1}{p}, \qquad \text{(P.4.8)}$$

where $K_{U_{21}}(p)$ - operational function (operational voltage transfer coefficient).

In fig.P.4.8 we designate nodes 1, 2 and bases node 0.

Let's compile matrix of node voltage (MNV) for nodes 1, 2

$$\Delta = \begin{bmatrix} \frac{1}{r} & -\frac{1}{r} \\ -\frac{1}{r} & \frac{1}{r}+\frac{1}{pL}+\frac{1}{r} \end{bmatrix}.$$

Operational voltage transfer coefficient

$$K_{U_{21}}(p) = \frac{\Delta_{12}}{\Delta_{11}};$$

$$\Delta_{12} = \frac{1}{r}; \qquad \Delta_{11} = \frac{1}{r}+\frac{1}{pL}+\frac{1}{r} = \frac{2}{r}+\frac{1}{pL}.$$

Then

$$K_{U_{21}}(p) = \frac{1}{r\left(\frac{2}{r}+\frac{1}{pL}\right)} = \frac{1}{2}\cdot\frac{p}{p+\frac{r}{2L}}. \qquad \text{(P.4.9)}$$

Now, accounting (P.4.8) , (P.4.9), we get

$$h_{K_U}(p) = \frac{1}{2}\cdot\frac{p}{p+\frac{r}{2L}}\cdot\frac{1}{p} = \frac{1}{2}\cdot\frac{1}{p+\frac{r}{2L}}. \qquad \text{(P.4.10)}$$

Original of ( P.4.10)

$$h_{K_U}(t) = \frac{1}{2}e^{-\frac{t}{\frac{2L}{r}}} = \frac{1}{2}e^{-\frac{r}{2L}t}. \qquad \text{(P.4.11)}$$

In fig. P.4.9 input signal assume meanings

$$\begin{cases} u_1(t) = 0 \text{ at } t < 0; \\ u_1(t) = A + \dfrac{B-A}{t_1} \text{ at } 0 \le t < t_1; \\ u_1(t) = B \text{ at } t_1 \le t < t_2; \\ u_1(t) = 0 \text{ at } t > t_2. \end{cases} \qquad \text{(P.4.12)}$$

At $t < 0$ output voltage $u_2(t) = 0$, as input action $u_1(t) = 0$. At section $0 \le t < t_1$ voltage $u_2(t)$ can be find in according Duhamel integral formula

$$f_{out}(t) = f_{in}(0)h(t) + \int_0^t f'_{in}(\tau)h(t-\tau)d\tau. \qquad \text{(P.4.13)}$$

Here, accounted (P.4.12), (P.4.13) we get at this section

$$\begin{cases} f_{in}(0) = u_{11}(0) = A; \\ h(t) = h_{K_U}(t) = \dfrac{1}{2}e^{-\frac{r}{2L}t}; \\ h(t-\tau) = h_{K_U}(t-\tau) = \dfrac{1}{2}e^{-\frac{r}{2L}(t-\tau)}; \\ f_{in}(t) = u_{11}(t) = A + \dfrac{B-A}{t_1}t; \\ f_{in}(\tau) = u_{11}(\tau) = \dfrac{B-A}{t_1}\tau. \end{cases} \qquad \text{(P.4.14)}$$

Accounted (P.4.14), we get from (P.4.13) at this section

$$u_2(t) = u_{11}(0)h_{K_U}(t) + \int_0^t u'_{11}(\tau)h_{K_U}(t-\tau)\,d\tau =$$

$$= \frac{A}{2}e^{-\frac{r}{2L}t} + \frac{(B-A)L}{-rt_1}e^{-\frac{r}{2L}t}\left(e^{-\frac{r}{2L}t} - 1\right) =$$

$$= \frac{A}{2}e^{-\frac{r}{2L}t} + \frac{(B-A)L}{-rt_1}e^{-\frac{r}{2L}t} + \frac{(B-A)L}{-rt_1} =$$

$$= \left[\frac{A}{2} + \frac{(B-A)L}{-rt_1}\right]e^{-\frac{r}{2L}t} + \frac{(B-A)L}{-rt_1}. \qquad \text{(P.4.15)}$$

At section $t_1 \le t < t_2$ integration interval from 0 to $t$ is divided at two parts: from 0 to $t_1$ and from $t_1$ to $t$. Then in according (P.4.13) we get for this interval

$$u_2(t) = u_{11}(0)h_{K_U}(t) + \int_0^{t_1} u'_{11}(\tau)h_{K_U}(t-\tau)\,d\tau +$$

$$+ \int_{t_1}^{t} u'_{12}(\tau)h_{K_U}(t-\tau)\,d\tau. \qquad \text{(P.4.16)}$$

Here $u_{12}(t) = B; u_{12}(\tau) = B; u'_{12}(\tau) = 0$.

Then at section $t_1 \le t < t_2$, accounted (P.4.15) and the second integral in (P.4.16) equal to 0 $[u_{12}(\tau) = 0]$, we get

$$u_2(t) = \frac{A}{2}e^{-\frac{r}{2L}t} + \frac{(B-A)L}{rt_1}e^{-\frac{r}{2L}t}\left(e^{-\frac{r}{2L}t} - 1\right) =$$

$$= \left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t} - 1\right)\right]e^{-\frac{r}{2L}t}. \qquad \text{(P.4.17)}$$

At section $t > t_2$ integrated interval from 0 to $t$ is divided at through sections: from 0 to $t_1$, from $t_1$ to $t_2$ and from $t_2$ to $t$. Then in according (P.4.13) we get at this interval

$$u_2(t) = u_{11}(0)h_{K_U}(t) + \int_0^{t_1} u'_{11}(\tau)h_{K_U}(t-\tau)\,d\tau +$$

$$+ \int_{t_1}^{t_2} u'_{12}(\tau)h_{K_U}(t-\tau)\,d\tau - u_{12}(t_2)h_{K_U}(t-t_2) + \qquad \text{(P.4.18)}$$

$$+ \int_{t_2}^{t} u'_{13}(\tau)h_{K_U}(t-\tau)\,d\tau.$$

Here second integral is equal zero, as $u_{12}(\tau) = 0$; $u_{12}(t_2) = B$; $h_{K_U}(t-t_2) = \frac{1}{2}e^{-\frac{r}{2l}(t-t_2)}$. Third integral is equal to zero too, as $u_{13}(\mathrm{t}) = 0$ at this interval, where from $u_{13}(\tau) = 0$. There for at section $t > t_2$ we get from (P.4.18), accounted (P.4.17)

$$u_2(t) = \left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t} - 1\right)\right]e^{-\frac{r}{2L}t} - B\frac{1}{2}e^{-\frac{r}{2L}(t-t_2)} =$$

$$= \left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t} - 1\right)\right]e^{-\frac{r}{2L}t} - \frac{B}{2}e^{-\frac{r}{2L}t}e^{-\frac{r}{2L}t_2} =$$

$$\left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t} - 1\right) - \frac{B}{2}e^{-\frac{r}{2L}t_2}\right]e^{-\frac{r}{2L}t}. \qquad \text{(P.4.19)}$$

Therefore, output voltage in fig. 4.8 by input impulse action are:

at section $t < 0$

$$u_2(t) = 0;$$

at section $0 \le t < t_1$

$$u_2(t) = \left[\frac{A}{2} + \frac{(B-A)L}{-rt_1}\right] e^{-\frac{r}{2L}t} + \frac{(B-A)L}{-rt_1};$$

at section $t_1 \le t < t_2$

$$u_2(t) = \left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t} - 1\right)\right] e^{-\frac{r}{2L}t};$$

at section $t > t_2$

$$u_2(t) = \left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t} - 1\right) - \frac{B}{2} e^{-\frac{r}{2L}t_2}\right] e^{-\frac{r}{2L}t}$$

Lets check the solution, using other formula Duhamel integral

$$f_{out}(t) = \int_0^t f_{in}(\tau) a(t-\tau) d\tau.$$

Let's find impulse characteristic $a_{K_U}(t)$ for the circuit (fig. P.4.8). In operational form

$$a_{K_U}(p) = K_{U_{21}}(p).$$

Using (P.4.9), we get

$$a_{K_U}(p) = K_{U_{21}}(p) = \frac{1}{2} \cdot \frac{p}{p + \frac{r}{2L}} = \frac{1}{2} \cdot \frac{p + \frac{r}{2L} - \frac{r}{2L}}{p + \frac{r}{2L}} =$$

$$= \frac{1}{2}\left(1 - \frac{r}{2L} \cdot \frac{1}{p + \frac{r}{2L}}\right).$$

Now original in according expansion formula

$$a_{K_U}(t) = \frac{1}{2}\left[\delta(t) - \frac{r}{2L} e^{-\frac{r}{2L}t}\right].$$

At $t < 0$ its evidence, $u_2(t) = 0$.

At section $0 \le t < t_1$ we get

$$\begin{cases} a(t-\tau) = a_{K_U}(p) = \frac{1}{2}\left[\delta(t-\tau) - \frac{r}{2L} e^{-\frac{r}{2L}(t-\tau)}\right]; \\ f_{in}(t) = u_{11}(t) = A + \frac{B-A}{t_1} t;\ f_{in}(\tau) = u_{11}(\tau) = A + \frac{B-A}{t_1} \tau. \end{cases}$$

Now

$$u_2(t) = \int_0^t u_{11}(\tau) a_{K_U}(t-\tau)\, d\tau =$$

$$= \int_0^t \left(A + \frac{B-A}{t_1}\tau\right) \cdot \frac{1}{2}\left[\delta(t-\tau) - \frac{r}{2L} e^{-\frac{r}{2L}(t-\tau)}\ \right] d\tau =$$

$$= \frac{A}{2}\int_0^t \delta(t-\tau)d\tau - \frac{Ar}{4L} e^{-\frac{r}{2L}t} \int_0^t e^{-\frac{r}{2L}\tau} d\tau +$$

$$+ \frac{B-A}{2t_1}\int_0^t \tau\delta(t-\tau)d\tau - \frac{(B-A)r}{4Lt_1} e^{-\frac{r}{2L}t} \int_0^t \tau e^{-\frac{r}{2L}\tau} d\tau.$$

Here

$$\int_0^t \delta(t-\tau)d\tau = 1; \ \int_0^t e^{-\frac{r}{2L}\tau} d\tau = \frac{1}{\frac{r}{2L}} e^{-\frac{r}{2L}\tau}\Big|_0^t = \frac{2L}{r}\left(e^{-\frac{r}{2L}t} - 1\right);$$

$$\int_0^t \tau\delta(t-\tau)d\tau = t; \quad \int_0^t \tau e^{-\frac{r}{2L}\tau} d\tau = \frac{2L}{r}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t\right)\cdot e^{-\frac{r}{2L}t}\right].$$

Then

$$u_2(t) = \frac{A}{2} - \frac{Ar}{4L} e^{-\frac{r}{2L}t} \frac{2L}{r}\left(e^{-\frac{r}{2L}t} - 1\right) + \frac{B-A}{2t_1}t -$$

$$- \frac{(B-A)r}{4Lt_1} e^{-\frac{r}{2L}t} \frac{2L}{r}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t\right)\cdot e^{-\frac{r}{2L}t}\right] =$$

$$= \frac{A}{2} - \frac{A}{2}\cdot\left(1 - e^{-\frac{r}{2L}t}\right) + \frac{B-A}{2t_1}t -$$

(P.4.20)

$$- \frac{(B-A)}{2t_1}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t\right)\cdot e^{-\frac{r}{2L}t}\right] e^{-\frac{r}{2L}t} =$$

$$= \frac{A}{2} e^{-\frac{r}{2L}t} + \frac{B-A}{2t_1}t - \frac{(B-A)L}{rt_1} e^{-\frac{r}{2L}t} + \frac{(B-A)L}{rt_1} - \frac{B-A}{2t_1}t =$$

$$= \left[\frac{A}{2} - \frac{(B-A)L}{rt_1}\right]\cdot e^{-\frac{r}{2L}t} + \frac{(B-A)L}{rt_1},$$

That is way result (P.4.20) at the section $0 \le t < t_1$ is coincided which (P.4.15).

At section $t_1 \le t < t_2$ we get

$$u_2(t) = \int_0^{t_1} u_{11}(\tau) a_{K_U}(t-\tau)\, d\tau + \int_{t_1}^{t} u_{12}(\tau) a_{K_U}(t-\tau)\, d\tau =$$

$$= \int_0^{t_1} \left(A + \frac{B-A}{t_1}\tau\right) \cdot \frac{1}{2}\left[\delta(t-\tau) - \frac{r}{2L} e^{-\frac{r}{2L}(t-\tau)}\ \right] d\tau +$$

$$+ \int_{t_1}^{t} B \cdot \frac{1}{2}\left[\delta(t-\tau) - \frac{r}{2L} e^{-\frac{r}{2L}(t-\tau)}\ \right] d\tau =$$

$$= \frac{A}{2}\int_0^{t_1} \delta(t-\tau) d\tau - \frac{Ar}{4L} e^{-\frac{r}{2L}t} \int_0^{t_1} e^{-\frac{r}{2L}\tau} d\tau + \frac{B-A}{2t_1}\int_0^{t_1} \tau\delta(t-\tau) d\tau -$$

$$- \frac{(B-A)r}{4Lt_1} e^{-\frac{r}{2L}t} \int_0^{t_1} \tau e^{-\frac{r}{2L}\tau} d\tau + \frac{B}{2}\int_{t_1}^{t} \delta(t-\tau) d\tau - \frac{Br}{4L}\int_{t_1}^{t} e^{-\frac{r}{2L}\tau} d\tau.$$

Here

$$\int_0^{t_1} \delta(t-\tau) d\tau = 0;\ \int_0^{t_1} e^{-\frac{r}{2L}\tau} d\tau = \frac{2L}{r}\left(e^{-\frac{r}{2L}t} - 1\right);$$

$$\int_0^{t_1} \tau\delta(t-\tau) d\tau = 0;\ \int_0^{t_1} \tau e^{-\frac{r}{2L}\tau} d\tau = \frac{2L}{r}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t_1\right) \cdot e^{-\frac{r}{2L}t_1}\right];$$

$$\int_{t_1}^{t} \delta(t-\tau) d\tau = 1;\ \int_{t_1}^{t} e^{-\frac{r}{2L}\tau} d\tau = \frac{2L}{r}\left(e^{-\frac{r}{2L}t} - e^{-\frac{r}{2L}t_1}\right).$$

Then

$$u_2(t) = -\frac{(B-A)}{2t_1} e^{-\frac{r}{2L}t}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t_1\right) \cdot e^{-\frac{r}{2L}t_1}\right] -$$

$$- \frac{B}{2} e^{-\frac{r}{2L}t_2} e^{-\frac{r}{2L}t} + \frac{B}{2} e^{-\frac{r}{2L}t_1} e^{-\frac{r}{2L}t} = \tag{P.4.21}$$

$$= \frac{A}{2} e^{-\frac{r}{2L}t} - \frac{A}{2} e^{-\frac{r}{2L}t_1} - \frac{(B-A)L}{rt_1} e^{-\frac{r}{2L}t} + \frac{(B-A)}{2t_1} \times$$

$$\times \left(\frac{2L}{r} - t_1\right) \cdot e^{-\frac{r}{2L}t_1} e^{-\frac{r}{2L}t} - \frac{B}{2} e^{-\frac{r}{2L}t_2} e^{-\frac{r}{2L}t} + \frac{B}{2} e^{-\frac{r}{2L}t_1} e^{-\frac{r}{2L}t} =$$

$$= \left[\frac{A}{2} - \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t_1} - 1\right) - \frac{B}{2}e^{-\frac{r}{2L}t_2}\right] \cdot e^{-\frac{r}{2L}t}.$$

That is way result (4.21) at the section $t_1 \le t < t_2$ is coincided which (P.4.15)

At section $t > t_2$ we get

$$u_2(t) = \int_0^{t_1} u_{11}(\tau) a_{K_U}(t-\tau)\, d\tau + \int_{t_1}^{t_2} u_{12}(\tau) a_{K_U}(t-\tau)\, d\tau +$$

$$+ \int_{t_2}^{t} u_{13}(\tau) a_{K_U}(t-\tau)\, d\tau =$$

$$= \int_0^{t_1} \left(A + \frac{B-A}{t_1}\tau\right) \cdot \frac{1}{2}\left[\delta(t-\tau) - \frac{r}{2L}e^{-\frac{r}{2L}(t-\tau)}\right] d\tau +$$

$$+ \int_{t_1}^{t_2} B \cdot \frac{1}{2}\left[\delta(t-\tau) - \frac{r}{2L}e^{-\frac{r}{2L}(t-\tau)}\right] d\tau.$$

Here, as $u_{13}(\tau) = 0$, then we get

$$u_2(t) = \frac{A}{2}\int_0^{t_1} \delta(t-\tau) d\tau - \frac{Ar}{4L}e^{-\frac{r}{2L}t}\int_0^{t_1} e^{-\frac{r}{2L}\tau} d\tau +$$

$$+ \frac{B-A}{2t_1}\int_0^{t_1} \tau\delta(t-\tau) d\tau - \frac{(B-A)r}{4Lt_1}e^{-\frac{r}{2L}t}\int_0^{t_1} \tau e^{-\frac{r}{2L}\tau} d\tau +$$

$$+ \frac{B}{2}\int_{t_1}^{t_2} \delta(t-\tau) d\tau - \frac{Br}{4L}e^{-\frac{r}{2L}t}\int_{t_1}^{t_2} e^{-\frac{r}{2L}\tau} d\tau.$$

where

$$\int_0^{t_1} \delta(t-\tau) d\tau = 0; \quad \int_0^{t_1} e^{-\frac{r}{2L}\tau} d\tau = \frac{2L}{r}\left(e^{-\frac{r}{2L}t_1} - 1\right);$$

$$\int_0^{t_1} \tau\delta(t-\tau) d\tau = 0; \quad \int_0^{t_1} \tau e^{-\frac{r}{2L}\tau} d\tau = \frac{2L}{r}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t_1\right) \cdot e^{-\frac{r}{2L}t_1}\right];$$

$$\int_{t_1}^{t_2} \delta(t-\tau)d\tau = 0; \int_{t_1}^{t_2} e^{-\frac{r}{2L}\tau}d\tau = \frac{2L}{r}\left(e^{-\frac{r}{2L}t_2} - e^{-\frac{r}{2L}t_1}\right).$$

Then

$$u_2(t) = -\frac{Ar}{4L}e^{-\frac{r}{2L}t}\frac{2L}{r}\left(e^{-\frac{r}{2L}t_1} - 1\right) -$$
$$\frac{B-A}{2t_1}e^{-\frac{r}{2L}t}\left[\frac{2L}{r} - \left(\frac{2L}{r} - t_1\right)\cdot e^{-\frac{r}{2L}t_1}\right] -$$
$$-\frac{B}{2}e^{-\frac{r}{2L}t} + \frac{B}{2}\left(e^{-\frac{r}{2L}t_1} - e^{-\frac{r}{2L}t}\right) =$$
$$= \frac{A}{2}e^{-\frac{r}{2L}t} - \frac{A}{2}e^{-\frac{r}{2L}t_1}e^{-\frac{r}{2L}t} - \frac{(B-A)L}{rt_1}e^{-\frac{r}{2L}t} +$$
$$= \frac{B-A}{2t_1}\left(\frac{2L}{r} - t_1\right)e^{-\frac{r}{2L}t_1}e^{-\frac{r}{2L}t} - \frac{B}{2}e^{-\frac{r}{2L}t_2}e^{-\frac{r}{2L}t} +$$
$$+\frac{B}{2}e^{-\frac{r}{2L}t_1}e^{-\frac{r}{2L}t} = \left[\frac{A}{2} + \frac{(B-A)L}{rt_1}\left(e^{-\frac{r}{2L}t_1} - 1\right) - \frac{B}{2}e^{-\frac{r}{2L}t_2}\right]e^{-\frac{r}{2L}t}. \tag{P.4.22}$$

That is way result (P.4.22) at the section $t > t_2$ is coincided which (P.4.19)

Consequently output voltage in the circuit of fig.P.4.8 if input signal is changed in fig. P.4.7 has the next form:

$t < 0,\ u_2(t) = 0;$

$0 \le t < t_1,\ u_2(t) = \left[\frac{A}{2} - \frac{(B-A)L}{rt_1}\right]e^{-\frac{r}{2L}t} + \frac{(B-A)L}{rt_1};$

$t_1 \le t < t_2, u_2(t) = \left[\frac{A}{2} - \frac{(B-A)L}{rt_1} + \frac{(B-A)L}{rt_1}e^{-\frac{r}{2L}t_1}\right]\cdot e^{-\frac{r}{2L}t};$

$t > t_2, u_2(t) = \left[\frac{A}{2} - \frac{(B-A)L}{rt_1} + \frac{(B-A)L}{rt_1}e^{-\frac{r}{2L}t_1} - \frac{B}{2}e^{-\frac{r}{2L}t_2}\right]\cdot e^{-\frac{r}{2L}t}$

Graphics of voltage $u_2(t)$ is shown in fig.P.4.10

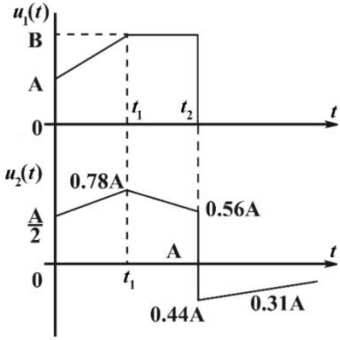


Fig.P.4.10

### 4.5. The convolution integral for envelope curves

In radio engineering the tasks of transient processes study in high-frequency oscillation circuits under the modulation high-frequency oscillations (Fig. 4.16) action on them are often found.

These processes are described by the

$$f_{in}(t) = F_{in}(t)\cos[\omega_c t + \psi_{in}(t)], \tag{4.53}$$

where Fin(t) and $\psi_{in}(t)$ are the amplitude and initial phase of high frequency oscillation; $\omega_c$ is the cyclic frequency of carrier high frequency oscillation. Functions Fin(t), $\psi_{in}(t)$ slowly vary in time. Function $F_{in}(t)$ is the envelope curve of the high-frequency oscillation $f_{in}(t)$.

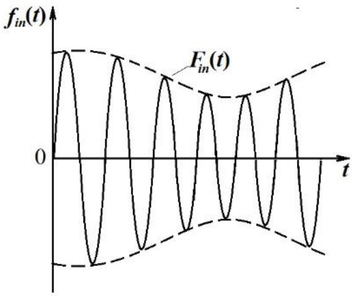


Fig. 4.16

The calculation of the transition process is simplified, if we confine ourselves by describing only the envelop curve of high-frequency oscillations.

Let's on a resonant contour acts the signal describing by the expression (4.53). To define the reaction $f_{out}(t)$.

To use the formula of the Duamel integral (4.21). Assume that the impulse characteristic of the resonant circuit has the form

$$a(t) = A(t)\cos(\omega_f t + \varphi_f), \tag{4.54}$$

where $A(t)$ is an envelop curve; $\omega_f$ is the own frequency of oscillation contour; $\varphi_f$ of is initial phase of free oscillations.

To substitute formulas (4.53.) and (4.54) in the expression (4.21):

$$f_{out}(t) = \int_0^t f_{in}(\tau)a(t-\tau)d\tau = \tag{4.55}$$

$$= \int_0^t F_{in}(\tau)\cos[\omega_c\tau + \psi_{in}(\tau)]A(t-\tau)\cos[\omega_f(t-\tau) + \varphi_f]\,d\tau.$$

To convert the expression (4.55) into the cosines product:

$$f_{out}(t) = \frac{1}{2}\int_0^t F_{in}(\tau)\cos[(\omega_c - \omega_f)\tau + \omega_f t + \psi_{in}(\tau) + \varphi_f] \times$$

$$\times A(t-\tau)d\tau + \tag{4.56}$$

$$+\frac{1}{2}\int_0^t F_{in}(\tau)\cos[(\omega_c + \omega_f)\tau - \omega_f t + \psi_{in}(\tau) - \varphi_f]A(t-\tau)d\tau.$$

The second integral in expression (4.56) is close to zero, since the integration is performed for a high frequency signal $(\omega_c + \omega_f)$. The area of the positive and negative half-waves of which are mutually destroyed on the interval of integration.

Тому

$$f_{out}(t) \approx \frac{1}{2}\int_0^t F_{in}(\tau)\,A(t-\tau)\cos[(\omega_c - \omega_f)\tau +$$

$$+\omega_f t + \psi_{in}(\tau) + \varphi_f]d\tau. \tag{4.57}$$

We can write the expression (4.57) through the instantaneous complex values:

$$f_{out}(t) \approx \frac{1}{2}\int_0^t F_{in}(\tau)\, A(t-\tau)\mathrm{Re}\left[e^{j(\omega_c-\omega_f)\tau}e^{j\psi_{in}(\tau)}\right]d\tau =$$

$$\mathrm{Re}\left[\frac{1}{2}\int_0^t F_{in}(\tau)\, A(t-\tau)e^{j\Delta\omega\tau}d\tau\right], \qquad (4.58)$$

where $F_{in}(\tau) = F_{in}(\tau)e^{j\psi_{in}(\tau)}$ is the complex envelop curve of input signal; $\Delta\omega = \omega_c - \omega_f$ is the absolute contour disorder.

From the expression (4.58) the complex envelop curve of reaction is

$$F_{out}(t) = F_{out}(t)e^{j\psi_{out}(t)} = \frac{1}{2}\int_0^t F_{in}(\tau)\, A(t-\tau)e^{j\Delta\omega\tau}d\tau.$$

If the phase of the input signal $\psi_{in}(\tau) = \mathrm{const}$ and absolute contour disorder $\Delta\omega \approx 0$, then

$$F_{out}(t) = \frac{1}{2}\int_0^t F_{in}(\tau)\, A(t-\tau)e^{j\Delta\omega\tau}d\tau. \qquad (4.59)$$

Expression (4.59) is a convolution of the envelop curves of the input signal and impulse characteristic of the electric circuit and it is called the convolution integral for the envelop curves.

**Example 4.5.**

To calculate the envelop curve of the electric current and $i(t)$ in the sequential oscillation circuit (Fig. 4.11), when it is switched by the harmonic voltage, which envelop curve is a stepwise function (Fig. 4.17,a):

$$u(t) = U_m \cdot 1(t).$$

If the input action is the voltage and the electric current is the reaction, then the impulse characteristic should be taken as impulse conductivity $a_Y(t)$. In the example 4.2 for a circuit (Fig. 4.11) it was determined by the formula

$$a_Y(t) = \frac{\omega_0}{\omega_f L}e^{-\delta t}\cos\left(\omega_f + \frac{\pi}{2} - \alpha\right).$$

Obviously, that its envelop curve is written as $A_Y(t) = \frac{\omega_0}{\omega_f L}e^{-\delta t}$.

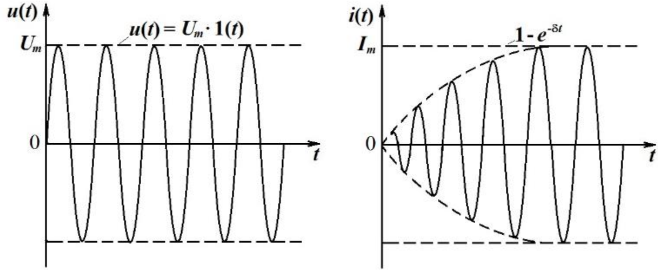


Fig. 4.17

Then by the formula (4.59) the envelop curve of electric current is expressed as

$$i(t) = \frac{1}{2}\int_0^t u(\tau)A_Y(t-\tau)d\tau = \frac{1}{2}\int_0^t U_m \cdot 1(t)\frac{\omega_0}{\omega_f L}e^{-\delta(t-\tau)}d\tau =$$

$$= \frac{\omega_0}{2\omega_f L}U_m e^{-\delta t}\int_0^t e^{\delta\tau}d\tau = \frac{\omega_0}{2\omega_f L\delta}U_m e^{-\delta t}e^{\delta\tau}\Big|_0^t =$$

$$= \frac{\omega_0}{2\delta\omega_f L}U_m\left(1-e^{-\delta t}\right).$$

Graph of the electric current i(t) is shown in Fig.4.17,b, where

$$I_m = \frac{\omega_0}{2\delta\omega_f L}U_m.$$

Consequently, the amplitude of the electric current oscillation in the contour increases smoothly.

## Methodic instruction

By study material of section “Method of convolution integral” to begin with acquiring essence of superposition in electrical circuit and order its application. Impotent role have circuit time characteristics – reaction of the circuit to standard pulse influence. It’s necessary distinguish transient and pulse characteristics, in spite of its dimensions (dimension of pulse characteristic is equal to dimension of transient characteristic, divided by second). The convolution integral is used for calculation transient processes.

Special attention it’s necessary to convolution integral by calculation passing signal of complicated forms through electrical circuits. The input action is divided on separate intervals. By that circuit reaction at any time moment on a given interval is equal to reaction on this interval plus reaction of the circuit on input signal, which action at all moment on previous intervals.

Literature: [1] - [5]; [7]; [9]; [10]; [14 - 16]

## Questions for self checking

1. What are circuit operational functions? What are varieties of them?

2. What is connected circuit operational function with circuit complex function?

3. Give an example of transient processes calculation with help circuit operational function.

4. Explain sense of superposition method in transient processes theory.

5. Determine standard test influence and connection between of them.

6. What of time characteristic are you known?

7. Show order of transient processes in electric circuit calculates.

8. What particularity of convolution method calculates if input action has gap of the first and second kind?

# 5. METHODS OF TRANSIENT PROCESSES ANALYSES IN THE NONLINEAR CIRCUITS

## 5.1. Particularity of transient processes in nonlinear circuits

Transient processes in nonlinear circuits are write down by nonlinear diferential equations, which haven't common solution methods. Character of these equations dependents on input voltage, and superposition principle isn't used. That is way, standard test signals, reactions on which are complete definite dynamic property of the circuit for example, unit step function $1(t)$ or impulse function $\delta(t)$ for nonliear circuits. Transfer operation function $H(p)$ and frequency characteristic $H(j\omega)$ aren't are absent too.

At the same time transient processes in nonlinear circuits are more diverse, then in linear circuits and corresponded peculiarity are used for working out of different elektrotechnical devices, which can be realizes in linear circuit impossible.

Using of the different methods dependence on peculiarity of concrete problem and on level of computer technique, which is can be used worker.

It is necessary to see, in nonlinear circuits the physical processes have be privies analyzed before calculation.

## 5.2. Integrate method of approximation

Integrate method of approximation is used if it's possible to pick up approximate analytical expression for nonlinearity in a given problem, which permit to compile differential equation for solution in analytical form. It is possible seldom and for the equations of non high order.

Let's consider example of transient processes in circuit for fig. 5.1, a, where constant voltage includes to series connected nonlinear two-port $NT(r)$ and inductive coil. Transient process is wrote down by differential equation

$$u_L + u_r + u_{NT} = L\frac{di}{dt} + ri + f(i) = U. \quad (5.1)$$

Let’s characteristic $i = f(u_{NT})$ (fig. 5.1,*b*) can be approximate on the same interval be parabola of the second order $i = au_{NT}^2$ or $u_{NT} = \sqrt{\frac{i}{a}}$. Then differential equation becomes in form

$$L\frac{d}{dt}(au^2) + rau^2 + u = U \tag{5.2}$$

or

$$2Lau\frac{du}{dt} + rau^2 + u = U \tag{5.3}$$

whence after division of variable we get

$$t = -2La\int_0^U \frac{u}{rau^2 + u - U}du =$$

$$= \frac{L}{r}(\ln \frac{U}{U - ri - \sqrt{\frac{i}{a}}} + \frac{1}{\Delta}\ln\frac{2r\sqrt{at} + 1 - \Delta}{2r\sqrt{at} + 1 + \Delta} - \mathrm{M}), \tag{5.4}$$

where $\Delta = \sqrt{(4raU + 1)}$; $\mathrm{M} = \frac{1}{\Delta}\ln\frac{1-\Delta}{1+\Delta}$.

Function $t = f(i)$ can’t be represent as open function $i = \varphi(t)$, therefore for construction graphic $i = \varphi(t)$ its necessary to give the several meaning $i$ and define accordance meaning $t$.

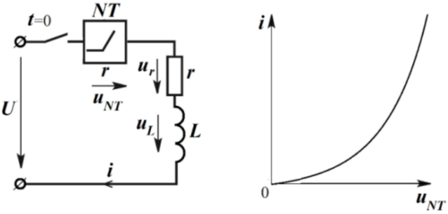
### 5.3. Graphic integration method

Graphic integration methods are enough labor-consuming and used for comparatively simple problem, for example, for calculate circuits, which are describe differential equations with division of variable. Here it is possible construction of a function graphic, curve of which limits area, which proportional accordance meaning of time.

Let’s consider application of this method for fig. 5.1, *a*.

Let’s divide variable in equation (5.1)

$$dt = L\frac{1}{U - rif(i)}di. \tag{5.5}$$




a b

Fig. 5.1. Switching *rL*-circuit witch nonlinear two-port:
*a* – circuit, *b* – V.A.C. of two-port

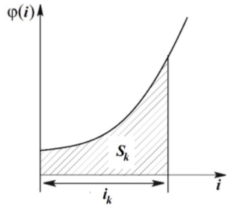



Fig. 5.2. Performance of graphic integration method
for calculate of the circuit

Let's integrate left and right parts by accordance variables. Then accounting zero initial conditions, we get

$$t = \int_0^t dt = \int_0^i \frac{L}{U - ri - f(i)} di. \quad (5.6)$$

Let's construct graphic of dependence

$$\varphi(i) = \frac{L}{U - ri - f(i)}. \quad (5.7)$$

From these graphic of area (regarding scale) we can find dependence $t = f(i)$ or $i = \varphi(t)$ (current $i$ in fig. 5.2).

### 5.4. Method of phase plane

By investigation of transient processes in nonlinear electrical circuits usually dependence of its parameters on times and in accordance of these by construct graphics time $t$ is lay out along abscess axis and investigate parameters: current, voltage, value of charge – along ordinate axes. But can be lay out investigate parameters ($i$, $u$, $Q$) along abscess axis and along ordinate axes – velocity of these values in time ($di/dt$, $du/dt$, $dQ/dt$).

Coordinate plane, on which along one axes (usually along abscess axis) investigation value $x$ is lay out, and along other axes (usually along ordinate axis) – velocity of these values in time $y = dx/dt$ is cold phase plane. That is way time is absent on the graphic, but graphic gives the full information about process.

Transient process on complex plane are imaged by the same strait or curve, if it describe by differential equation no more second order. Method of phase plane isn't used practically for circuit of more second order.

Change state of system can be image by motion the same point on complex plane. These pointer is cold «representation» or «production». Co-ordinates of representation pointer $x$ and $y$ define its position on phase plane and characterizes state of process at a given moment time. At time representation pointer moves and describe the line, which is cold "phase trajectory". Kind of phase trajectory depends on circuit and its parameters.

By periodic process phase trajectory is closed circle (for linear circuit that is circumference or ellipse), which representation pointer describe during each period. Phase trajectory for the none periodic process is not circumference line.

In the upper half plain derivative of coordinate $y > 0$, hence representation pointer can be move only to the right – in direction increasing meaning $x$. In the lower half plain $y < 0$, representation pointer can be move only to the left. Consequently representation pointer move only clockwise direction. Dependence from initial condition we get difference phase trajectory, which never intersect. On the abscise excise $dx/dt = 0$, then phase trajectory cross these excise under right angle.

Family of phase trajectory, which images processes in a given circuit is cold "phase portrait". Phase portrait allow envelope all totality of move in a system, which may be arose in considered system. Conclusion about moves may be without advance founding analytical expressions integral of initial equations even and then, when these expressions can't be received (that is very important).

Points of phase plane, where simultaneously $dx/dt = 0$ and $dy/dt = 0$, are called "special points". They correspond balance conditions (immobility) of considered circuit and may be steady and non steady.

Special point, through which doesn't though one phase trajectory and which is surrounded by closed trajectories, is named «centre». Centre corresponds regime of irreversible balance.

Special point, which is asymptotical for the phase trajectories, is named «focus». Focus is named «resistant», if image point approach to them, "non resistant", - if one move away.

Special point, through which phase trajectories move, is named «knot». If move along phase trajectories has direction to knot, then such knot is named "resistant", if move along phase trajectories has direction from knot, - "non resistant".

For transient processes (oscillate, aperiodic e.t.c.) in linear circuit of the first and second order are the phase portraits, witch which can be compare phase portrait of investigatory circuit. For the some nonlinear circuit their phase portraits, but number of varieties of such circuit are more great. Therefore their phase portraits are very difficulty.

Phase portraits are compile, as a rule, for the circuit without energy source, but in the same case can be receive phase portrait in the frost regime.

For construct graphic of dependence $i(t)$ it's necessary to define the time moments, which correspond to pointers of phase trajectory. Time interval $t$, during of which transition is accomplished from $k$-th point ($x_k$, $y_k$) of phase trajectory to the close ($k$+1)-th point ($x_k$, $y_{k+1}$), can approximate calculate by the next in the same way. As $y = dx/dt$, then

$$\Delta t = \int_{x_k}^{x_{k+1}} \frac{1}{y} dx.$$

Let's mark $\frac{1}{y} = f(x)$. In according with "theorem about average" we get

$$\Delta t = f(\varepsilon)(x_{k+1} - x_k) = f(\varepsilon)\Delta x, \ (x_k < \varepsilon < x_{k+1}).$$

By little interval $\Delta t$ and monotonous changed y in this interval may receiver

$$f(\varepsilon) \approx \frac{1}{y_{av}},$$

where

$$y_{av} = \frac{y + y_{k+1}}{2}.$$

Then

$$\Delta t \approx \frac{x}{y_{av}}.$$

In fig. 5.3 several phase portraits for linear circuit (free running) and special points and graphics of dependences $x = f(t)$ (for one phase trajectories), which define accordance phase portraits, are shows.

Let's conceder circuit fig. 5.1, a and construct phase trajectory of transient processes for its (accounted forts regime).

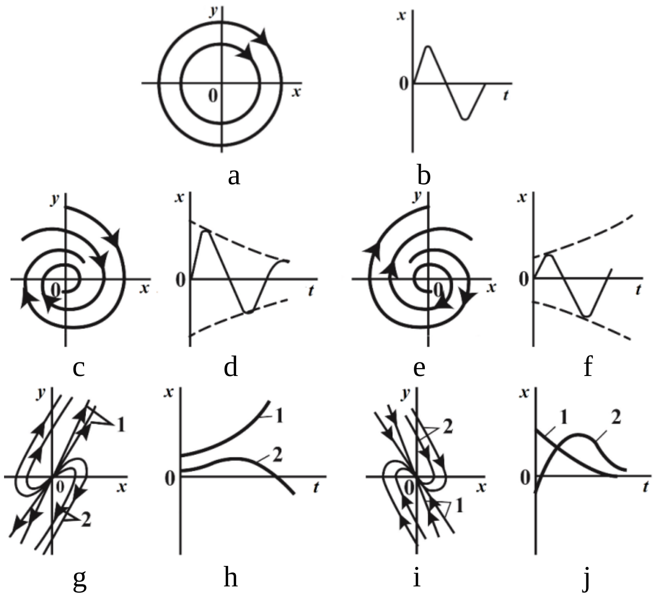



Fig. 5.3. Phase portraits and graphics of dependences $x(t)$ in linear circuits (free regime):
*a, b* – non subside oscillations; *c, d* – decreasing oscillations;
*i, f* – increasing oscillations; increasing aperiodic processes;
*g, h* – decreasing aperiodic processes

Let’s differential equation (5.1) in form

$$\frac{di}{dt} = \frac{1}{L[U - ri - f(i)]}, \tag{5.8}$$

Give different meaning of $i$, and find accordance meaning $di/dt$ (fig. 5.4). Giving graphic testifies about aperiodic increasing character of transient processe witch steady state meanings $di/dt = 0$, $i = I_y$.

Dependence $i(t)$ can be receive from fig. 5.4, if segment on abscissa from 0 to $I_y$ to divide on small intervals $\Delta i$, fined $y_{av} = (di/dt)_{av}$ and determine accordance meaning $\Delta t$.

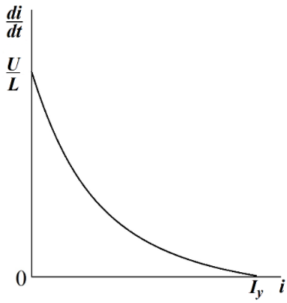



Fig. 5.4. Phase trajectory of transient processes in circuit fig. 5.1, *a*

### 5.5. Method of successive approximations

These method consist in successive closer definition privies receiving a different way initial approximation. Founding of this approximation is very cumbersome and difficult operation.

Let’s consider circuit of fig. 5.1, a. Let’s substitute nonlinear two-port NT ($r$) and linear resistance $r$ for equivalent nonlinear two-port, using method of summing up volt-ampere characteristic. Characteristic of equivalent nonlinear two-port $i(u_r + u_{NT})$ is showed in fig. 5.5, *b*.

Differential equation of these circuit

$$u_L + u_{rNT} = L\frac{di}{dt} + f_1(i) = U \tag{5.9}$$

is nonlinear, but it is possible in the first approximation to laniaries it by means substitution curve $i(u_r + u_{NT})$ by strait line $i'(u)$ (fig. 5.5, *b*).

This line goes through beginning of coordinates and point on characteristic $i(u_r + u_{NT})$, which corresponds of study state conditions. All resistances are linear in these regime then study state mining

$$= \frac{U}{r_{1st}} = \frac{U}{r + r_{NTst}}, \tag{5.10}$$

where

$$r_{1st} = \text{tg}\,\alpha.$$

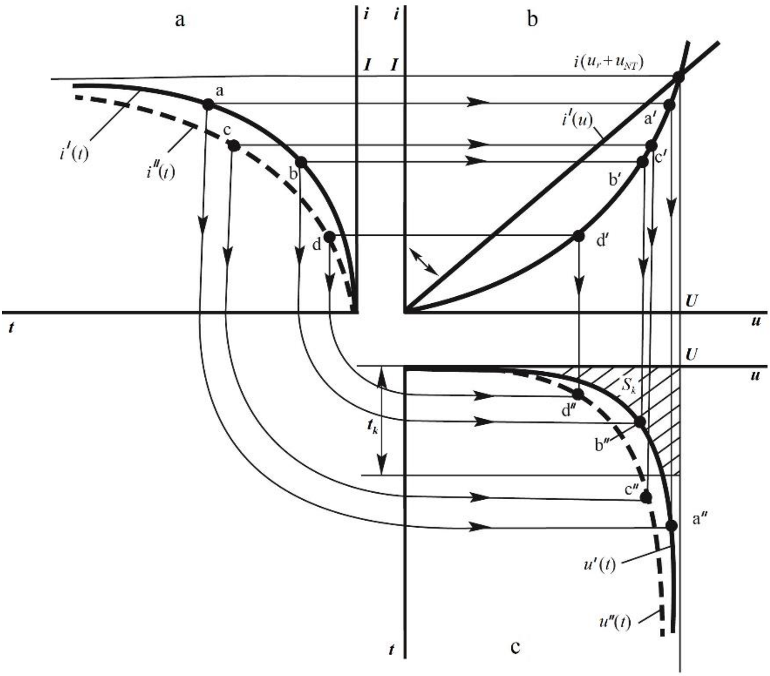



Fig. 5.5 Method of successive approximations:
$a$ – graphics of dependences $i(t)$;
$b$ – VAC $i(u)$; $c$ - graphics of dependences $u(t)$

Differential equation for such circuit is linear

$$L\frac{di'}{dt} + r_{1st}i' = U. \tag{5.11}$$

Solution of these equation

$$i' = I\left(1 - e^{-\frac{r_{1st}}{L}t}\right). \tag{5.12}$$

Let’s construct graphic of dependence $i'(t)$ for linear circuit (fig. 5.5, a). Let’s used nonlinear characteristic $i(u_r + u_{NT})$ for founding points of curve $u'(t)$. Graphic construction of dependence $u'(t)$ is shown in fig. 5.5 for two points ($a$ and $b$). That is way, we perform the first stage of closer definition solution. Here we used non strait line $i'(t)$ (see fig. 5.5, $b$), but initial nonlinear characteristic $i(u_r + u_{NT})$.

Graphic of dependence $u'(t)$ is constructed for next closer definition of solution. Using expression (5,9), we get

$$i = \frac{1}{L}\int_0^t (U - u)\, dt. \tag{5.13}$$

That allows to use method of graphic integration and find correspond current $i_k$ (fig. 5.5, c) for the arbitrary time moment $t_k$ (with accounting of scale)

$$i_k = \frac{S_k}{L}. \tag{5.14}$$

Assuming different times moments $t_1$, $t_2$ e.g., may by find meaning of currents $i_1$, $i_2$ e.t.c. and construct in fig. 5.5, a new curve $i''(t)$, which exactly shows express dependence current from time, then approximate function (5.12). Using this curve and nonlinear characteristic $i(u_r + u_{NT})$ we can construct dependence $u''(t)$, as it shows in fig. 5.5 for two pointes (*c* and *d*).

Further again assume meanings of time and more accurate dependence $i(t)$. It is necessary to note this demands verification on convergence.

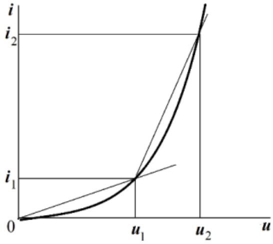
## 5.6. Mating intervals method

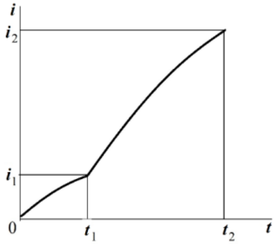
Idea of mating intervals method consist in breaking the process on series following one after the other intervals, inside of which transient process may be exactly or approximately writing down by linear or integrate nonlinear differential equation. Integrate constants in this equation are define from limit conditions, accounting demand of solution continuity (this operation is cold mating).

Mating intervals method is universal, but calculation becomes cumbersome for the circuit of the high order and by long during transient processes and by necessary high exactly be means decreasing of select intervals. By calculation can by computers but method of final intervals is more comfortable.

Let's conceder mating intervals method for the circuit fig. 5.1, a.

Let's divide volt-ampere characteristic on section for the piece ways linear approximation (fig. 5.6, a). Then:




a b

Fig. 5.6. Mating intervals method:
$a$ – piece – wise-linear approximation of VAC;
$b$ – graphic of dependence $i(t)$

For the first section

$$i = \frac{i_1}{u_1} = \frac{u}{r_1};$$

$$r_1 = \frac{u_1}{i_1}; \tag{5.15}$$

$$u = r_1 i. \tag{5.16}$$

For the second section

$$i = i_1 + \frac{i_2 - i_1}{u_2 - u_1}(u - u_1) = i_1 + \frac{u - u_1}{r_2};$$

$$r_2 = \frac{u_2 - u_1}{i_2 - i_1}; \tag{5.17}$$

$$u = r_2 i + u_1\left(1 - \frac{r_2}{r_1}\right). \tag{5.18}$$

Using receiving meanings, we get instead equation (5.1) two linear equations:

For the first section

$$r_1 i + ri + \frac{di}{dt} = E = E_1 \text{ by } 0 \le i \le i_1; \;\; 0 \le t \le t_1. \tag{5.19}$$

For the second section

$$r_2 i + ri + L\frac{di}{dt} = E - u_1\left(1 - \frac{r_2}{r_1}\right) = E_2 \qquad (5.20)$$

by $i_1 \le i \le i_2$; $t_1 \le t \le t_2$.

Solutions of differential equation for these sections are:

$$i = \frac{E_1}{r_1 + r} + A_1 e^{-\frac{r_1 + r}{L}t} \quad \text{by } 0 \le t \le t_1; \qquad (5.21)$$

$$i = \frac{E_2}{r_2 + r} + A_2 e^{-\frac{r_1 + r}{L}(t - t_1)} \quad \text{by } t_1 \le t \le t_2. \qquad (5.22)$$

Integration constant $A_1$ is find from condition, that by $t = 0$, $i = 0$

$$A_1 = -\frac{E_1}{r_1 + r}$$

Then for the first section

$$i = \frac{E_1}{r_1 + r}\left(1 - e^{-\frac{r_1 + r}{L}t}\right), \text{ by } 0 \le t \le t_1. \qquad (5.23)$$

Substitution in this equation $i = i_1$ and $t = t_1$ gives

$$i_1 = \frac{E_1}{r_1 + r}\left(1 - e^{-\frac{r_1 + r}{L}t_1}\right), \qquad (5.24)$$

where find time moment

$$t_1 = -\frac{L}{r_1 + r}\ln\left(1 - \frac{r_1 + r}{E_1}i_1\right). \qquad (5.25)$$

Integration constant $A_2$ is find from equation for the second section.

By $t = t_1$ and $i = i_1$

$$A_2 = i_1 - \frac{E_2}{r_2 + r}.$$

Equation for the second section

$$i = \frac{E_2}{r_2 + r} + \left(i_1 - \frac{E_2}{r_2 + r}\right) e^{-\frac{r_1 + r}{L}(t - t_1)}, \text{ by } t_1 \le t \le t_2. \quad (5.26)$$

Time moment $t_2$ is find from condition, that by $i = i_2$, $t = t_2$. Than

$$i_2 = \frac{E_2}{r_2 + r} + \left(i_1 - \frac{E_2}{r_2 + r}\right) e^{-\frac{r_1 + r}{L}(t_2 - t_1)}, \qquad (5.27)$$

where from we find time moment $t_2$:

$$t_2 = t_1 - \frac{L}{r_2 + r}\ln\left[1 + \frac{r_2 + r}{E_2}(i_1 - i_2)\right]. \qquad (5.28)$$

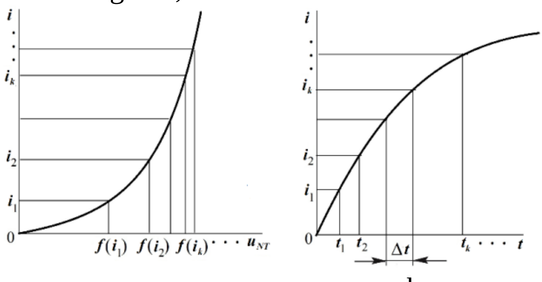
Using equations for the different sections we can build dependence $i = \varphi(t)$ kind of which is shown in fig. 5.6, *b*.

## 5.7. Fined increment method (of successive sections)

Given method is more common method (numerical integration), but it demands the large work expenditure. Time interval are divided into enough little time intervals $t$ ( integrate step) and differentials changed by final increments during of this time interval. Further transfer to the Taylor series for the solution of differential equation. As Taylor series is infinite then its necessary to limit the same numbers of its component. If the lower first component has second order, then method is name method by Euler.

Receiving pointers can be trace on graphic. If this pointers are strait line. Then this method is named Euler method. For more exactly solution equation between pointers may be method by Adams and method by Runge-Kutta. All these methods (Euler, Adams, Runge-Kutta) have general name "Fined increment method or method of successive sections".

Let's considere circuit in fig. 5.1, *a*. Characteristic of nonlinear two-port is shows in fig. 5.7, *a*.




a b

Fig. 5.7. Fined increment method:
*a* – VAC; *b* – graphic of dependence $i(t)$

Nonlinear differential equation (5.1) may be represented in form (5.8).

Let's change derivative $di/dt$ by relation of final increment. We get approximation equation

$$\Delta i \approx \frac{1}{L}[U - ri - f(i)]\Delta t, \qquad (5.29)$$

Let's divide transient processes time into row of small intervals $\Delta t$. Then we get for any ($k$+1)-th interval ($k$ = 0, 1,2 …)

$$\Delta i_{k+1} = i_{k+1} - i_k \approx \frac{1}{L}[U - ri_k - f(i_k)]\Delta t, \qquad (5.30)$$

where $i_k$ and $i_{k+1}$ – instantaneous meanings of current in beginning and in end of considered time interval.

That is way, it's possible step by step to calculate row of instantaneous meaning currents: using initial meaning current $i_0$ may by define current to the end of the first interval $i_1$and using meaning $i_1$ – current $i_2$ etc (fig. 5.7, *b*).

It's comfortable to perform calculate in table form, for example table 5.1.

Table 5.1

| $k$ | $t_k = k\Delta t$ | $i_k$ | $ri_k$ | $f(i_k)$ | $U - ri_k - f(i_k)$ | $\Delta i_{k+1}$ | $i_{k+1} = i_k + \Delta i_{k+1}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | $i_0$ | 0 | 0 | $U$ | $\Delta i_1$ | $i_1 = \Delta i_1$ |
| 1 | $t_1 = \Delta t$ | $i_1$ | $ri_1$ | $f(i_1)$ | $U - ri_1 - f(i_1)$ | $\Delta i_2$ | $i_2 = \Delta i_2$ |
| 2 | $t_2 = 2\Delta t$ | $i_2$ | $ri_2$ | $f(i_2)$ | $U - ri_2 - f(i_2)$ | $\Delta i_3$ | $i_3 = \Delta i_3$ |
| … | … | … | … | … | … | … | … |

Result of solution is more exactly (see fig. 5.7), when time interval $\Delta t$ is lesser. But if total number of intervals is increase the common errors of calculation is increase too. That is basic shortage of this method. Before named may be very little by using computers.

## 5.8. Method of state space

Differential equations by calculation transient processes in the nonlinear electrical circuits can be compile as in classical form and in form of "state equations".

Selecting of variable in Kirchhoff equations for nonlinear circuits have the same peculiarity. If circuit includes one or the same nonlinearity, then as variables comfortable to take not current and voltages but flax linkages and charges as in linear circuits because characteristics of nonlinear reactive two-ports are given in form weber-ampere or charge-voltage characteristic. Besides, if characteristics of nonlinear two-ports consists breaking off the first order, flax linkages and charges hasn't jumpers on the this section of characteristics.

Order of equations in the nonlinear circuit coincides witch quantity reactive linear and nonlinear reactive two-ports. For example, circuit witch one inductance is described by nonlinear differential equation of the first order

$$f_1\left(\frac{d\psi}{dt}, \psi, t\right) = 0. \tag{5.31}$$

This equation it's necessary by weber-ampere characteristic of two-port.

$$\Psi = \psi(i), \tag{5.32}$$

which may be given analytically, graphically or in form table.

Equation (5.31) can by ratting down in form state equations.

$$\frac{d\psi}{dt} = f_2(\psi, t). \tag{5.33}$$

Selection form of writing down for nonlinear differential equation defines by method of solution for that problem. Let's used method of numeral integration for calculate of state equations.

Let's considered circuit in fig. 5.8. Lets Weber-ampere characteristic $\psi = f_1(i)$ or $i = f_2(\psi)$. Choose flax linkage as state space $\psi$. Right down state equation

$$\frac{d\psi}{dt} = U - rf_2(\psi). \tag{5.34}$$

Solution of this equation

$$\Psi = \psi(0) + \int_0^t U dt - r\int_0^t f_2\,(\psi) dt. \tag{5.35}$$

Let's solve this equation by numeral method, taking step of integration $T$ current $i = f_2(\psi)$ as constant. Then for the same $k$-ht step of integration we get equation

$$\Psi[(k+1)T] = \psi(kT) + UT - ri(kT)T, \tag{5.36}$$

where $i(kT) = f_2[\psi(kT)]$.

From structure of this equation, for solution initial equation (5.34) was used method of finite increments (sequence intervals).

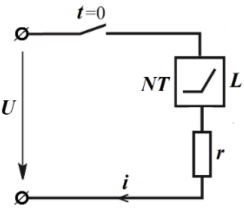


Fig. 5.8. Nonlinear circuit of the first order

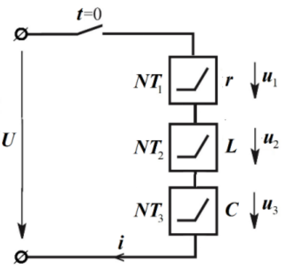


Fig. 5.9. Nonlinear circuit of the second order

Let's calculate transient processe in circuit of fig. 5.1, *a*, where nonlinear two-port have volt-ampere characteristic in fig. 5.1, *b*. Here as state space we may chose current $i$. Then state equations

$$\frac{di}{dt} = -\frac{r}{L}i + \frac{1}{L}U - \frac{1}{L}U_{NT}(i), \qquad (5.37)$$

where $U_{NT}(i)$ – is characteristic of nonlinear two-port.

Solution of equation (5.37) is

$$i = e^{-\frac{r}{L}t}\, i(0) + \frac{1}{L}\int_0^t e^{-\frac{r}{L}(t-\tau)}\, U d\tau - \frac{1}{L}\int_0^t e^{-\frac{r}{L}(t-\tau)}\, U_{NT}(i) d\tau, \qquad (5.38)$$

where $i(0)$ – initial condition.

Assuming the same assumption (constancy voltage on nonlinear two-port of integrate step), we get rated formula

$$i[(k+1)T] = \frac{e^{-\frac{r}{L}T}}{i(kT)} + \frac{1}{L}\left(e^{-\frac{r}{L}T} - 1\right)\left(-\frac{L}{r}\right)U -$$
$$-\frac{1}{L}\left(e^{-\frac{r}{L}T} - 1\right)\left(-\frac{L}{r}\right)U_{NT}(kT). \qquad (5.39)$$

The circuit of the second order describes by two nonlinear differential equation of the first order or one equation of the second order. Let's in circuit all three to ports are nonlinear. Their characteristics are:

$$U = f_1(i) \text{ or } i = f_2(U_1); \tag{5.40}$$

$$\Psi = f_3(i) \text{ or } i = f_4(\psi); \tag{5.41}$$

$$Q = f_5(u_3) \text{ or } u_3 = f_6(Q). \tag{5.42}$$

For this circuit may by right down the next equation

$$\frac{dQ}{dt} = f_4(\psi); \tag{5.43}$$

$$f_1(i) + \frac{d\psi}{dt} + f_6(Q) = U. \tag{5.44}$$

This system can be lead to one nonlinear differential equation of the second order. Differentiating equation (5.44) (in according rules of differentiation of no open function) and to take the placing (5.43), we get

$$\frac{d^2\psi}{dt^2} + \left[\frac{df_1}{di}\frac{df_4}{d\psi}\right]\frac{d\psi}{dt} + \frac{d\psi_6}{dQ}f_4(\psi) = U. \tag{5.45}$$

This system may be writing down in form state equations

$$\frac{dQ}{dt} = f_4\,(\psi); \quad \frac{d\psi}{dt} = -f_1[f_4(\psi)] - f_6(Q) + U. \tag{5.46}$$

Circuit including, $n$ nonlinear reactive two-ports, is rout down in common with help system of nonlinear differential equations of the first order. If variables are state variables, then state equations of the circuit in matrix form are rout down as

$$d\boldsymbol{X}/dt = \boldsymbol{A}(\boldsymbol{X})\boldsymbol{X} + \boldsymbol{B}\boldsymbol{W}, \tag{5.47}$$

where $\boldsymbol{X}$ – state vector of $n$-th order, which includes currents of inductive two-ports and voltages of capacitance two-ports (or flax linkages and changes); $\boldsymbol{A}(\boldsymbol{X})$ – matrix of the coefficients with seize $n \times n$, elements of which depends on state variables; $\boldsymbol{B}$ – matrix of the constant coefficients with seize $m \times n$ ($m$ – number of voltages and currents sources); $\boldsymbol{W}$ – vector of voltages and currents sources.

### 5.9. Methods of averaging

Methods of averaging are based on assumption, which not always possible: parameter of the circuit is changed very small and therefore it may be assume constant on this interval and equals its averaging meaning.

Primitive variant of such method is first approach, where value of impedance of nonlinear two-port is assumed constant during all transient processe. But this impedance is defined very rough, since averaging is performed only between initial and final meanings.

These methods are more develop for alternative and common periodic currents, where any circuit coordinate (envelope of sinusoid amplitude, constant component) is changed low and may be assume as constant value.

By slow changing of amplitude envelope allows to perform variant of averaging methods, which is cold as method of slowly changing amplitude. Mathematic operations of this method reduce to presentation of envelope in form harmonic series and equation of the circuit is integrated in bound of period. Then all harmonic component gives zero and only enough simple approximate solution, which corresponds to the constant components and which showing the changing of basic (first) harmonic amplitude in transient processe (without components).

For circuit of periodic current with constant component the correspond variant of averaging method (method of slowly changing of constant component) gives approximation solution for constant component (without alternative component).

As example let's considered the problem.

**Problem 5.5**

Capacitor which capacitance $C = 100$ mcF is charged to voltage $U = 40$ V. Define current by discharge this capacitor on nonlinear resistance two-port, characteristic of which is given in table 5.4.

Table 5.4

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $U$ | V | 0 | 5 | 10 | 20 | 30 | 40 |
| $I$ | A | 0 | 0,05 | 0,11 | 0,22 | 0,295 | 0,33 |

*Solution*.

Let's we used method of successive approximations for solution.

Let's linearize of volt-ampere characteristic of nonlinear two-port in accordance fig. 5.10, *b*.

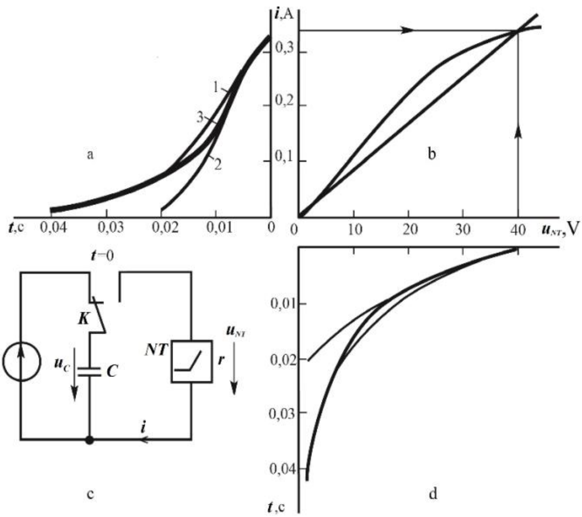



Fig. 5.10

$a$ – graphics of the dependence $i(t)$; $b$ – VAC and its linearity;
$c$ – circuit; $d$ – graphics of the dependences $u(t)$

Find dependence $u_{NT}(t) = u_r(t)$ for linear circuit. It's known, solution of equation for $rC$ – circuit in free regime is

$$u_{\,rf} = Ae^{-pt} = Ae^{-\frac{t}{\tau}},$$

where $r = 40/0.33 = 121.2$ Ohms; $rC = 121.2 \cdot 100 \cdot 10 = 0.012$ s.

Using independent initial conditions $u_{\mathrm{c}}(0) = 40$ V and equation of the circuit $u_C + u_k = 0$, find (account , the forced component of voltage equals zero)

$$u_r = -40e^{\,-83{,}3t}\text{V},$$

or for comfortable construction

$$u_r = 40e^{-83{,}3t}\text{V}.$$

Let's construct voltage $u_{NT}(t)$ in fig. 5.10, $d$ (curve *1*) using dates of table 5.5 (first approximation).

Then voltage $u_{NT}(t)$ is carried on VAC (fig. 5.10, $b$). We get the first approximation of current $i(t)$ (curve 1. fig. 5.10, $a$).

As $u_c = u_{NT} = \frac{1}{\mathrm{C}} \int_0^t idt$ (on absolute value) it is possible fined second approximation of voltage $u_{\mathrm{NT}}(t)$,if calculate area under curve $i(t)$.

Table 5.5

| t,s | The first approximation | | The second approximation | | | The third approximation | | |
|---|---|---|---|---|---|---|---|---|
| | u, V | i, A | S, $mm^2$ | u, V | i, A | S, $mm^2$ | u, V | i, A |
| 0 | 40 | 0,33 | 0 | 40 | 0,33 | 0 | 40 | 0,33 |
| 0,005 | 26,5 | 0,275 | 600 | 25 | 0,265 | 590 | 25,3 | 0,27 |
| 0,01 | 17,5 | 0,195 | 1040 | 14 | 0,13 | 1011 | 14/7 | 0,165 |
| 0,015 | 11,5 | 0,125 | 1360 | 6 | 0,065 | 1200 | 10 | 0,11 |
| 0,02 | 7,5 | 0,08 | 1540 | 1,5 | 0,015 | 1320 | 7 | 0,075 |
| 0,03 | 3,3 | 0,033 | - | - | - | - | 3,3 | 0,033 |
| 0,04 | 1,5 | 0,015 | - | - | - | - | 1,5 | 0,015 |

Then account non zero initial conditions we get

$$u_{NT}(t) = u_{NT}(0) - \frac{1}{C} mS = \frac{5 \cdot 10^{-3} \cdot 5 \cdot 10^{-4}}{(100 \cdot 10^{-6})} S = 25 \cdot 10^{-3} S, \text{ V},$$

where $m$ – scale along access current and time.

During second approximation of the voltage (curve *2*, fig. 5.10, *d*) we found second approximation of the of the current (curve *2*, fig. 5.10, *a*), and then the three approximation of the voltage (curve *3*, fig. 5.10, *d*) and three approximation of the of the current (curve *3*,fig. 5.10, *a*). Results calculation are given in table 5.5.

**Problem 5.6.**

Series connection of inductance $L = 0.08$ H and nonlinear resistance two-port, characteristic of which is given in table 5.3, are connected join in constant voltage $U = 40$ V. Define current in the circuit. It's necessary the problem by method of successive sections.

*Solution.*

Let's we substitute in differential equation of the circuit (fig. 5.11, *a*)

$$L\frac{di}{dt} + u = U$$

derivative by relations of finite increments and receive approximation equation

$$\Delta i \approx \frac{U - u_1}{L} \Delta t.$$

Let’s divide transient processe time on rang of small same intervals $\Delta t = 0.1$mcs. Then for the any ($k$-th) interval we get

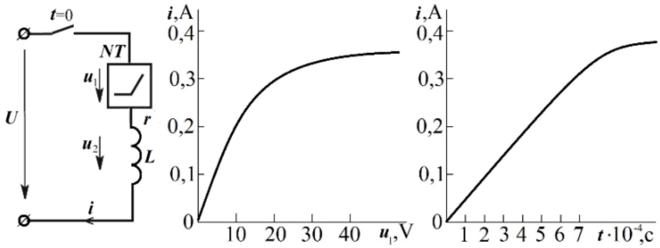
$$i_{k+1} = i_k + \Delta i_{k+1} = \frac{U - u_1}{L}\Delta t\,.$$

Using volt-ampere characteristic of nonlinear two-port (fig. 5.11, $b$), we perform calculate and compile tabl. 5.6.

Table 5.6

| № | $t_k$, ms | $i_k$, A | $u_{1k}$, V | $U-u_{1k}$, V | $\frac{U-u_{1k}}{L}$, V/H | $\Delta i_{k+1}$, A | $i_{k+1}$, A |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 40 | 500 | 0.05 | 0.05 |
| 1 | 0.1 | 0.05 | 1.5 | 38.5 | 481 | 0.0481 | 0.0981 |
| 2 | 0.2 | 0.0981 | 3 | 37 | 462 | 0.0462 | 0.1443 |
| 3 | 0.3 | 0.1443 | 4.75 | 35.25 | 440 | 0.044 | 0.1883 |
| 4 | 0.4 | 0.1883 | 7 | 33 | 412 | 0.0412 | 0.2295 |
| 5 | 0.5 | 0.2295 | 10 | 30 | 375 | 0.0375 | 0.267 |
| 6 | 0.6 | 0.267 | 14.5 | 25.5 | 320 | 0.032 | 0.299 |
| 7 | 0.7 | 0.299 | 20 | 20 | 250 | 0.025 | 0.324 |
| 8 | 0.8 | 0.326 | 35 | 5 | 62 | 0.0062 | 0.33 |
| 9 | 0.9 | 0.33 | - | - | - | - | - |

Dependance $i(t)$ is shown in fig. 5.11, $c$.




a b c

Fig. 5.11.

$a$ – circuit; $b$ – VAC; $c$ – graphic of dependence $i(t)$

**Problem 5.7**.

Series connection of resistance $r = 10$ Ohms and nonlinear inductive two-port are connected to source of constant voltage $U = 1$ V. Define current of the circuit, if characteristic of nonlinear two-port is wrote down by equation $i = \psi^2$. Solve problem by method of state space.

*Solution*.

Scheme of the circuit is shown in fig. 5.8 and is circuitscribed by equation (5. 34)

$$\frac{d\psi}{dt} = U - rf_2(\psi, t). \qquad (5.34)$$

Solution of this equation is (5.36)

$$\Psi[(k+1)T] = \psi(kT) + UT - ri(kT)T, \qquad (5.36)$$

where $i(kT) = f_2[\psi(kT)]$.

Calculate meaning of dependences $\psi(t)$ and $i(t)$ are given in table 5.8, in accordance of which are construct graphics of fig. 5.12.

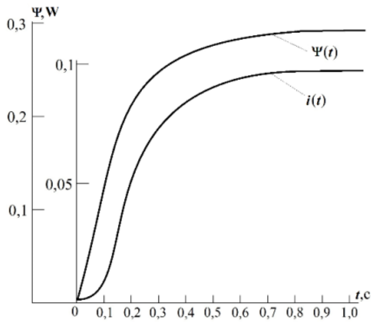



Fig. 5.12. Graphics of dependences $\Psi(t)$ and $i(t)$

Table 5.8

| *K* | *t* = *kT*, s | Ψ(*kT*), W | *i*(*kT*), A |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0,1 | 0,1 | 0,01 |
| 2 | 0,2 | 0,19 | 0,036 |
| 3 | 0,3 | 0,254 | 0,0645 |
| 4 | 0,4 | 0,289 | 0,0835 |
| 5 | 0,5 | 0,305 | 0,093 |
| 6 | 0,6 | 0,312 | 0,097 |
| 7 | 0,7 | 0315 | 0,0992 |
| 8 | 0,8 | 0,3158 | 0,0997 |
| 9 | 0,9 | 0,3161 | 0,0999 |
| 10 | 1,0 | 0,3162 | 0,09998 |

## Methodic instruction

By analyze and calculation transient processes in nonlinear circuits its necessary to attention on certain particularity of this circuit.

1. Nonlinear differential equation, which describe processes in nonlinear circuits haven't common solutions even for circuits of the first and second order.
2. Superposition principle isn't applicable to the nonlinear circuits.
3. For nonlinear circuits standard test functions as single step function $1(t)$ and delta function $\delta(t)$, which completely define reaction of the circuit on such action, aren't existed.
4. Transference $H(p)$ and frequency $H(j\omega)$ functions aren't define property of nonlinear circuits.
5. Electro technical devices, constructed on nonlinear elements are more diverse, then on linear circuits. Nonlinear circuits allow construct such devices, realization of which in linear circuits isn't possible.
6. Offering methods of analyses and calculations of nonlinear circuits can't apply without preliminary understanding physical processes in the circuit.

Literature: [ 4, 9, 10, 12, 14,15, 18, 20]

## Questions for self checking

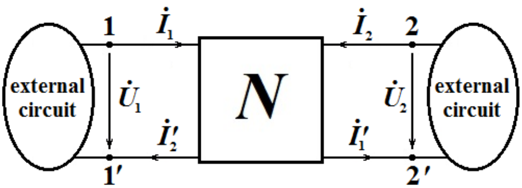
1. What is Integrate method of approximation?
2. What is Graphic integration method?
3. What is Method of phase plane?
4. What is Method of successive approximations?
5. What is Mating intervals method?
6. What is Fined increment method (of successive sections)?
7. What is Method of state space?
8. What is Methods of averaging

# 6. BESIS OF TWO-PORTS THEORY

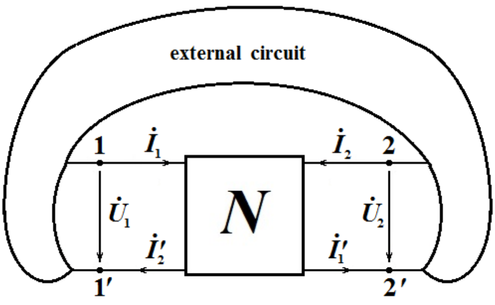
## 6.1.Basic notions and definitions

Electric circuits by the number of external terminals can be divide into two-port, three-port, four-port and multi-port.

A two-port (four-terminal network) is called a part of an electric circuit, that has two pairs of external terminals, through which it connects to the rest of the circuit. In fig. 6.1 two-port *N* with external terminals1-1', 2-2' connects to the other circuit.




a




b

Fig. 6.1

Two-ports are classified according to various features (fig. 6.2).

Linear are called two ports, which haven't nonlinear elements. The two-ports are called nonlinear, if they contain at least one nonlinear element.

Passive is called two-ports, which do not have sources of energy. The two-port is called active, if its composition includes at least one source of voltage or current. If all the energy sources that are part of the active two-ports are autonomous (uncontrolled), then the two-port is called autonomous. If at least one of them is non-autonomous (uncontrolled), then the two-port is called the non-autonomous.

Two-ports, which satisfy the principle of reciprocity, called reciprocal, otherwise – non-reciprocal.

Symmetric are called two-ports, whose circuits are symmetrical relative to the vertical, conducted through the middle of the circuits, asymmetric – otherwise.

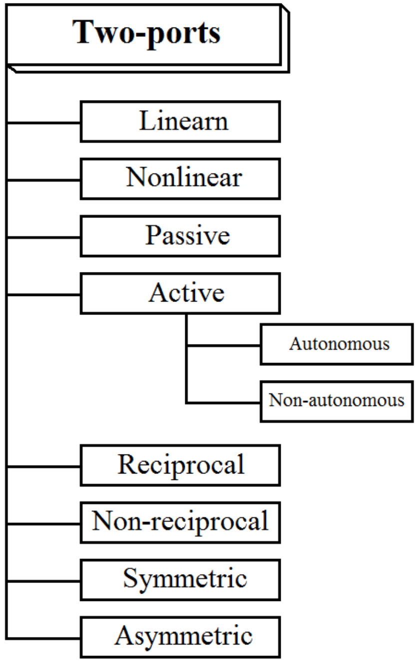
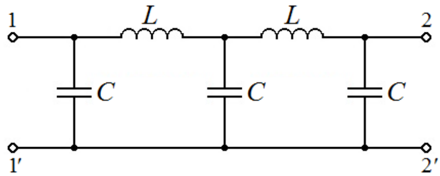



Fig. 6.2

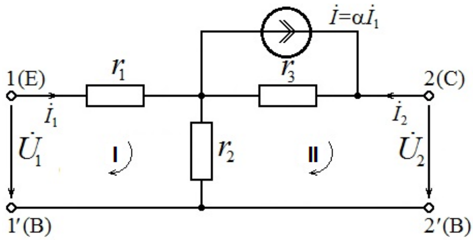
Fig. 6.3 shows the schematics of the simplest two-ports. Two-port, shown in Fig. 6.3, a (lower pass filter) is linear, passive, mutual and symmetric, the two-port, shown in fig. 6.3, b (scheme of replacement of the transistor with a source of current) – nonlinear, active, non-autonomous, asymmetric, and in fig. 6.3, c (circuit for replacing the transistor with a voltage source) – linear, active, autonomus, asymmetric. In fig. 6.3, d the outputs E, B, C corresponds to the emitter, base and collector of the transistor.

The main purpose of the two-port is the transmission of energy from the source to the consumer and the transformation of parameters: amplification, attenuation, stabilization, frequency transformation, phase, voltage, power, etc.

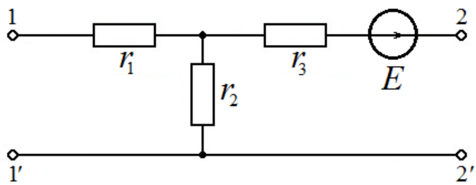
The main task of the two-port theory is to study its properties with respect to its external terminals as well as the analysis of the work of the two-port with the external circuits.




a




b




c

Fig. 6.3

The two-port with the outer circuit can be connected by regular and irregular connections. If the two-port connects two independent parts of the outer circuit (see Fig. 6.1, a), then it is considered passable and has a fair relation to it:

$$\dot I_1 = \dot I_1', \quad \dot I_2 = \dot I_2'.$$

Such a connection is called regular. If the two-port connects the areas of one outer circuit (Fig. 6.1,b), then in the general case, such a connection is called irregular

$$\dot I_1 \neq \dot I_1', \quad \dot I_2 \neq \dot I_2'.$$

The methods of the theory of two-ports are only related to regular connections. At the input and output of the two-port there are two voltages and two currents. The relations between them are described by a system of two equations, in which any variables from these can be considered as independent, and the other two as dependent. As a result, you can make $C_4^2 = 6$ pairs of equations, six systems of coefficients, which are called two-port parameters.

### 6.2. Two-ports equations

Let the two-port terminals 1-1' (Fig. 6.4) connect the source of the input electromotive force (EMF) with the internal impedance $\dot{E}_{inp}$, and to the terminals 2-2' the load $Z_l$.

Let's analyze the scheme of Fig. 6.4 by the method of loop currents.

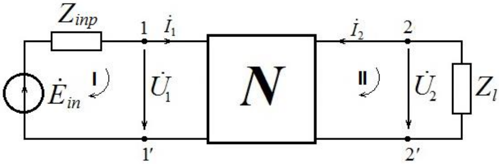



Fig. 6.4

Denote loop currents at the input and output of the two-port via $\dot{I}_{\text{I}}$, $\dot{I}_{\text{II}}$. We write the system of loop equations in a matrix form

$$\begin{bmatrix} Z_{inp} + Z'_{11} & Z'_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_l + Z'_{22} & \cdots & Z_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} \dot{I}_{\text{I}} \\ \dot{I}_{\text{II}} \\ \cdots \\ \dot{I}_N \end{bmatrix} = \begin{bmatrix} \dot{E}_{in} + \dot{E}'_{\text{I}} \\ \dot{E}_{\text{II}} \\ \cdots \\ \dot{E}_N \end{bmatrix}$$

where $Z'_{11}$, $Z'_{22}$ – the components of the complex impedance of the first and second loops, which are determined by the actual two- ports $N$ from the terminals 1-1' and 2-2' respectively; $\dot{E}'_{\text{I}}$ – component of the EMF of the first loop by the terminals 1-1' of the two-ports.

From Fig. 6.4 of two-ports is clear that.

$$\dot{I}_{\text{I}} = \dot{I}_1;\ \dot{I}_{\text{II}} = -\dot{I}_2;\ \dot{E}_{in} = \dot{I}_1 Z_{inp} + \dot{U}_1;\ \dot{U}_2 = -\dot{I}_2 Z_l. \quad (6.1)$$

Components

$$\dot{I}_{\text{I}} Z_{inp} = \dot{I}_1 Z_{inp} = \dot{E}_{in} - \dot{U}_1$$

in the right side of the first equation (6.1) and

$$\dot{I}_{\text{II}} Z_l = -\dot{I}_2 Z_l = -\dot{U}_2.$$

$\dot{E}_{in}$ in the left part of the second equation (6.1) we move to the right part. Then, taking into account the expression (6.1)

$$\begin{bmatrix} Z_{inp} & Z'_{12} & \cdots & Z_{1N} \\ Z_{21} & Z'_{22} & \cdots & Z_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \\ \cdots \\ \dot{I}_N \end{bmatrix} = \begin{bmatrix} \dot{E}'_{\mathrm{I}} + \dot{U}_1 \\ \dot{E}_{\mathrm{II}} - \dot{U}_2 \\ \cdots \\ \dot{E}_N \end{bmatrix} \qquad (6.2)$$

The loop impedance matrix (6.2) is composed only of the impedance of the actual two-port without taking into account the outer circuit.

We have from system (6.2)

$$\begin{cases} \dot{I}_1 = \dfrac{\Delta_1}{\Delta} = \left(\dot{E}'_{\mathrm{I}} + \dot{U}_1\right)\dfrac{\Delta_{11}}{\Delta} + \left(\dot{E}_{\mathrm{II}} - \dot{U}_2\right)\dfrac{\Delta_{21}}{\Delta} + \cdots + \dot{E}_N\dfrac{\Delta_{N1}}{\Delta}; \\ -\dot{I}_2 = \dfrac{\Delta_2}{\Delta} = \left(\dot{E}'_{\mathrm{I}} + \dot{U}_1\right)\dfrac{\Delta_{12}}{\Delta} + \left(\dot{E}_{\mathrm{II}} - \dot{U}_2\right)\dfrac{\Delta_{22}}{\Delta} + \cdots + \dot{E}_N\dfrac{\Delta_{N2}}{\Delta}. \end{cases} \qquad (6.3)$$

where $\Delta$, $\Delta_1$, $\Delta_2$, $\Delta_{11}$, …, $\Delta_{N1}$, $\Delta_{12}$, ..., $\Delta_{N2}$ – determinants and algebraic adjunct of the matrix of loop impedances in the system (6.2)

From formulas (6.3) it is clear that the currents at the input and output of the two port are determined by the EMF and the smpedances of not only their own but also the remaining loops of the two-ports. Let's denote

$$\begin{cases} \dot{I}_{10} = \dot{E}'_{\mathrm{I}}\dfrac{\Delta_{11}}{\Delta} + \dot{E}_{\mathrm{II}}\dfrac{\Delta_{21}}{\Delta} + \cdots + \dot{E}_N\dfrac{\Delta_{N1}}{\Delta}; \\ \dot{I}_{20} = \dot{E}'_{\mathrm{I}}\dfrac{\Delta_{12}}{\Delta} + \dot{E}_{\mathrm{II}}\dfrac{\Delta_{22}}{\Delta} + \cdots + \dot{E}_N\dfrac{\Delta_{N2}}{\Delta}. \end{cases}$$

The currents $\dot{I}_{10}$ and $\dot{I}_{20}$ are determined by independent sources of EMF $\dot{E}'_{\mathrm{I}}$, $\dot{E}_{\mathrm{II}}$, …, $\dot{E}_N$ within the of two-port, that is, the system (6.3) can be rewritten as

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} + \begin{bmatrix} \dot{I}_{10} \\ \dot{I}_{20} \end{bmatrix}, \qquad (6.4)$$

where

$$Y_{11} = \frac{\Delta_{11}}{\Delta}; Y_{12} = -\frac{\Delta_{21}}{\Delta}; Y_{21} = -\frac{\Delta_{12}}{\Delta}; Y_{22} = \frac{\Delta_{22}}{\Delta}. \qquad (6.5)$$

The coefficients $Y_{11}$, $Y_{12}$, $Y_{21}$, $Y_{22}$ are determined only by passive elements of the two-port circuit and are called the parameters of the two-port.

Equation (6.4) is the equation of arbitrary two-port.

## 6.3. Parameters of the two-port

*Y- parameters*. For a passive two-port

$$\dot{I}_{10} = \dot{I}_{20} = 0.$$

Then we find from the system (6.4)

$$\begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} \tag{6.6}$$

or

$$[\dot{I}] = [Y][\dot{U}], \tag{6.7}$$

where

$$[Y] = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{\Delta_{11}}{\Delta} & -\frac{\Delta_{21}}{\Delta} \\ -\frac{\Delta_{12}}{\Delta} & \frac{\Delta_{22}}{\Delta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \Delta_{11} & -\Delta_{21} \\ -\Delta_{12} & \Delta_{22} \end{bmatrix}. \tag{6.8}$$

Elements $Y_{11}$, $Y_{12}$, $Y_{21}$, $Y_{22}$ are conducts and are called *Y*-parameters of two-port. The matrix (6.8) is a matrix of *Y*-parameters.

The system (6.6) can be rewritten in the usual form:

$$\begin{cases} \dot{I}_1 = Y_{11}\dot{U}_1 + Y_{12}\dot{U}_2; \\ \dot{I}_2 = Y_{21}\dot{U}_1 + Y_{22}\dot{U}_2. \end{cases} \tag{6.9}$$

From here it is clear that

$$Y_{11} = \left.\frac{\dot{I}_1}{\dot{U}_1}\right|_{\dot{U}_2=0};\ Y_{12} = \left.\frac{\dot{I}_1}{\dot{U}_2}\right|_{\dot{U}_1=0};\ Y_{21} = \left.\frac{\dot{I}_2}{\dot{U}_1}\right|_{\dot{U}_2=0};\ Y_{22} = \left.\frac{\dot{I}_2}{\dot{U}_2}\right|_{\dot{U}_1=0}. \tag{6.10}$$

that is, the *Y*-parameters can be determined experimentally by performing a short circuit test at the input ($\dot{U}_1 = 0$) at the calculation $Y_{12}$ and $Y_{22}$ at the output ($\dot{U}_2 = 0$) when calculating $Y_{11}$, $Y_{21}$.

Therefore *Y*-parameters are called short circuit parameters.

For reciprocal two-ports $Y_{12} = Y_{21}$, ie $\Delta_{12} = \Delta_{21}$. For simetrical two-ports $\Delta_{11} = \Delta_{22}$.

**Example 6.1.**

Determine the *Y* - parameters of the two-port (Fig. 6.5).

1. *Calculation method*. Let's construct a matrix of the loop impedance for the first (I) and second (II) loops is shown in Fig. 6.5.

$$Z = \begin{bmatrix} Z_1 + Z_2 & -Z_2 \\ -Z_2 & Z_2 + Z_3 \end{bmatrix}. \tag{6.11}$$

From here

$$\Delta = (Z_1 + Z_2)(Z_2 + Z_3) - Z_2^2 = Z_1Z_2 + Z_1Z_3 + Z_2Z_3; \qquad (6.12)$$

$$\Delta_{11} = Z_2 + Z_3; \Delta_{12} = Z_2; \Delta_{21} = Z_2; \Delta_{22} = Z_1 + Z_2. \qquad (6.13)$$

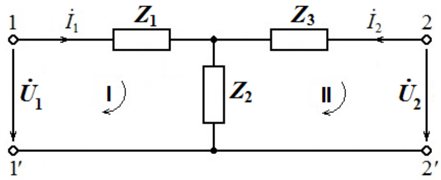



Fig.6.5

Now we obtain from formulas (6.5) taking into account (6.12), (6.13):

$$Y_{11} = \frac{\Delta_{11}}{\Delta} = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}; \qquad (6.14)$$

$$Y_{12} = \frac{\Delta_{21}}{\Delta} = -\frac{Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}; \qquad (6.15)$$

$$Y_{21} = \frac{\Delta_{12}}{\Delta} = -\frac{Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}; \qquad (6.16)$$

$$Y_{22} = \frac{\Delta_{22}}{\Delta} = \frac{Z_1 + Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}. \qquad (6.17)$$

2. *Experimental method*. Execute in the scheme of Fig. 6.5 short-circuit test on output terminals 2-2’ ($\dot U_2 = 0$):

$$\dot I_1 = \frac{\dot U_1}{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}}; \dot I_2 = -\frac{\dot I_1 Z_2}{Z_2+Z_3}.$$

Then

$$Y_{11} = \left.\frac{\dot I_1}{\dot U_1}\right|_{\dot U_2=0} = \frac{\dot U_1}{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}}\frac{1}{\dot U_1} = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3};$$

$$Y_{21} = \left.\frac{\dot I_2}{\dot U_1}\right|_{\dot U_2=0} = -\frac{\dot U_1}{Z_1 + \frac{Z_2Z_3}{Z_2+Z_3}}\frac{Z_2}{Z_2+Z_3}\frac{1}{\dot U_1} = -\frac{Z_2}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}.$$

We execute in the scheme of Fig. 6.5 a short-circuit test on the input terminals 1-1’ ($\dot U_1 = 0$):

$$\dot I_1 = -\frac{\dot I_2 Z_2}{Z_1+Z_2};\ \dot I_2 = \frac{\dot U_2}{Z_3 + \frac{Z_1 Z_2}{Z_1+Z_2}}.$$

Then

$$Y_{22} = \left.\frac{\dot I_2}{\dot U_2}\right|_{\dot U_1=0} = \frac{\dot U_2}{Z_3 + \frac{Z_1 Z_2}{Z_1+Z_2}}\frac{1}{\dot U_2} = \frac{Z_1+Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3};$$

$$Y_{12} = \left.\frac{\dot I_1}{\dot U_2}\right|_{\dot U_1=0} = -\frac{\dot U_2}{Z_3 + \frac{Z_1 Z_2}{Z_1+Z_2}}\frac{Z_2}{Z_1+Z_2}\frac{1}{\dot U_2} = -\frac{Z_2}{Z_1Z_2+Z_1Z_3+Z_2Z_3}.$$

*Z-parameters*. From expression (6.7)

$$[\dot U] = [Y]^{-1}[\dot I]$$

or

$$\begin{bmatrix}\dot U_1\\ \dot U_2\end{bmatrix} = \begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}^{-1}\begin{bmatrix}\dot I_1\\ \dot I_2\end{bmatrix}. \quad (6.18)$$

The inverse matrix has the form

$$[Y]^{-1} = \begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}^{-1} = \frac{1}{[Y]}\begin{bmatrix}Y_{11} & -Y_{12}\\ -Y_{21} & Y_{22}\end{bmatrix} = \begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix} = [Z], \quad (6.19)$$

where $[Y]$ – the determinant of the matrix *Y*-parameters from the formula (6.8)

$$[Y] = Y_{11}Y_{22} - Y_{12}Y_{21}. \quad (6.20)$$

The elements $Z_{11}$, $Z_{12}$, $Z_{21}$, $Z_{22}$ are impedances and are called Z-parameters of the two-port. The matrix (6.19) is a matrix of *Z*-parameters. Taking into account the relation (6.5) we obtain

$$[Z] = \begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix} = \frac{\Delta}{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}}\begin{bmatrix}\Delta_{22} & \Delta_{21}\\ \Delta_{12} & \Delta_{11}\end{bmatrix} =$$

$$= \frac{1}{\Delta_{11,22}}\begin{bmatrix}\Delta_{22} & \Delta_{21}\\ \Delta_{12} & \Delta_{11}\end{bmatrix} \quad (6.21)$$

The ratio is used here

$$\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21} = \Delta_{11,22},$$

where $\Delta_{11,22}$ – is the double algebraic adjunct, obtained from the determinant $\Delta$ by the striking out of the first and the other two rows and columns.

From expression (6.21)

$$Z_{11} = \frac{\Delta_{22}}{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}};\ Z_{12} = \frac{\Delta_{21}}{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}};$$
$$Z_{21} = \frac{\Delta_{12}}{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}};\ Z_{22} = \frac{\Delta_{11}}{\Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}}.$$

Thus, according to formulas (6.18) and (6.19), we obtain

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} \tag{6.22}$$

The system (6.22) can be rewritten in the usual form:

$$\begin{cases} \dot{U}_1 = Z_{11}\dot{I}_1 + Z_{12}\dot{I}_2; \\ \dot{U}_2 = Z_{21}\dot{I}_1 + Z_{22}\dot{I}_2. \end{cases} \tag{6.23}$$

From here it is clear that

$$Z_{11} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{I}_2=0};\ Z_{12} = \left.\frac{\dot{U}_1}{\dot{I}_2}\right|_{\dot{I}_1=0};\ Z_{21} = \left.\frac{\dot{U}_2}{\dot{I}_1}\right|_{\dot{I}_2=0};\ Z_{22} = \left.\frac{\dot{U}_2}{\dot{I}_2}\right|_{\dot{I}_1=0}, \tag{6.24}$$

ie $Z$-parameters can be determined experimentally, performing the experiment of without load at the input ($\dot{I}_1 = 0$) at the calculation $Z_{12}$ and $Z_{22}$ at the output ($\dot{I}_2 = 0$) at the calculation $Z_{11}$, $Z_{21}$. Therefore – Z-parameters are also called without load parameters.

We have from matrixes (6.19) and (6.21)

$$Z_{11} = \frac{\Delta_{22}}{\Delta_{11,22}} = \frac{Y_{22}}{Y_{11}Y_{22} - Y_{12}Y_{21}};\ Z_{12} = \frac{\Delta_{21}}{\Delta_{11,22}} = -\frac{Y_{12}}{Y_{11}Y_{22} - Y_{12}Y_{21}};$$
$$Z_{21} = \frac{\Delta_{12}}{\Delta_{11,22}} = -\frac{Y_{21}}{Y_{11}Y_{22} - Y_{12}Y_{21}};\ Z_{22} = \frac{\Delta_{11}}{\Delta_{11,22}} = \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}Y_{21}}.$$

For mutual two-ports $Z_{12} = Z_{21}$, ie $\Delta_{12} = \Delta_{21}$. For symmetrical two-ports, besides this $\Delta_{11} = \Delta_{22}$.

*Y*- and *Z*-parameters are dual. They are also called immittance parameters.

**Example 6.2**.

Determine the *Z*-parameters of the two-port (Fig. 6.5).

1. *Calculation method*. Matrix of the loop impedances is shown in (6.11). Definition of the matrix $[Y]$ according to the expressions (6.14) - (6.17) and (6.20) has the form

$$[Y] = Y_{11}Y_{22} - Y_{12}Y_{21} = \frac{Z_2+Z_3}{Z^2}\frac{Z_1+Z_2}{Z^2} - \frac{Z_2}{Z^2}\frac{Z_2}{Z^2} =$$

$$= \frac{(Z_2+Z_3)(Z_1+Z_2) - Z_2^2}{Z^4} = \frac{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}{Z^4} =$$

$$= \frac{1}{Z^2} = \frac{1}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}, \tag{6.25}$$

$$Z^2 = Z_1Z_2 + Z_1Z_3 + Z_2Z_3.$$

Using the expressions (6.14) - (6.17) and (6.25), we obtain:

$$Z_{11} = \frac{Y_{22}}{[Y]} = \frac{Z_1+Z_2}{Z^2}Z^2 = Z_1+Z_2;\ \ Z_{12} = -\frac{Y_{12}}{[Y]} = \frac{Z_2}{Z^2}Z^2 = Z_2;$$

$$Z_{21} = -\frac{Y_{21}}{[Y]} = \frac{Z_2}{Z^2}Z^2 = Z_2;\ \ Z_{22} = \frac{Y_{11}}{[Y]} = \frac{Z_2+Z_3}{Z^2}Z^2 = Z_2+Z_3.$$

2. *Experimental method*. Execute the without load on the output terminals 2-2' ($\dot{I}_2 = 0$) in the diagram shown in Fig.6.5. By that

$$\dot{I}_1 = \frac{\dot{U}_1}{Z_1+Z_2};\ \dot{U}_2 = \dot{I}_1 Z_2.$$

Than

$$Z_{11} = \frac{\dot{U}_1}{\dot{I}_1}\bigg|_{\dot{I}_2=0} = \frac{\dot{U}_1}{\frac{\dot{U}_1}{Z_1+Z_2}} = Z_1+Z_2;$$

$$Z_{21} = \frac{\dot{U}_2}{\dot{I}_1}\bigg|_{\dot{I}_2=0} = \frac{\dot{I}_1 Z_2}{\dot{I}_1} = Z_2.$$

Perform in the circuit in Fig. 6.5 without load at the input 1-1' ($\dot{I}_1 = 0$). With

$$\dot{I}_2 = \frac{\dot{U}_2}{Z_2+Z_3};\ \dot{U}_1 = \dot{I}_2 Z_2.$$

$$Z_{12} = \frac{\dot{U}_1}{\dot{I}_2}\bigg|_{\dot{I}_1=0} = \frac{\dot{I}_2 Z_2}{\dot{I}_2} = Z_2;\ \ Z_{22} = \frac{\dot{U}_2}{\dot{I}_2}\bigg|_{\dot{I}_1=0} = \frac{\dot{U}_2}{\frac{\dot{U}_2}{Z_2+Z_3}} = Z_2+Z_3.$$

*A-parameters*. In equations (6.6) for – the *Y*-parameters of independent variables are the voltages $\dot{U}_1$ and $\dot{U}_2$ at the input and output of the two-port. In equations (6.22) for *Z* - parameters independent variables are the currents $\dot{I}_1$ and $\dot{I}_2$ at the input and output of the two-port. We select independent alternating current $\dot{I}_2$ and voltage $\dot{U}_2$ at the output of the two-port. We obtain the voltage $\dot{U}_1$ and current $\dot{I}_1$ at the input of the two-port from the expression (6.6):

$$\begin{cases} \dot{U}_1 = -\dfrac{Y_{22}}{Y_{21}}\dot{U}_2 + \dfrac{1}{Y_{21}}\dot{I}_2 = A_{11}\dot{U}_2 - A_{12}\dot{I}_2; \\ \dot{I}_1 = -\dfrac{[Y]}{Y_{21}}\dot{U}_2 + \dfrac{Y_{11}}{Y_{21}}\dot{I}_2 = A_{21}\dot{U}_2 - A_{22}\dot{I}_2. \end{cases} \quad (6.26)$$

where

$$A_{11} = -\frac{Y_{22}}{Y_{21}};\ A_{12} = -\frac{1}{Y_{21}};\ A_{21} = -\frac{[Y]}{Y_{21}};\ A_{22} = -\frac{Y_{11}}{Y_{21}}.$$

The elements $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$ are complex transmitting functions of the two-port from output to input and called *A*-parameters of the two-port.

The expression (6.26) can be rewritten in the matrix form:

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ -\dot{I}_2 \end{bmatrix}$$

From expression (6.26) it is clear that

$$A_{11} = \left.\frac{\dot{U}_1}{\dot{U}_2}\right|_{\dot{I}_2=0};\ \ A_{12} = -\left.\frac{\dot{U}_1}{\dot{I}_2}\right|_{\dot{U}_2=0};$$
$$A_{21} = \left.\frac{\dot{I}_1}{\dot{U}_2}\right|_{\dot{I}_2=0};\ \ A_{22} = -\left.\frac{\dot{I}_1}{\dot{I}_2}\right|_{\dot{U}_2=0}. \quad (6.27)$$

That is, *A*-parameters can be determined experimentally by performing without load at the output ($\dot{I}_2 = 0$) when calculating $A_{11}$, $A_{21}$ and testing a short circuit at the output ($\dot{U}_2 = 0$) when calculating $A_{12}, A_{22}$.

By the physical nature the parameter $A_{11}$ is a complex transmission coefficient of voltage at no-load on the output, the parameter $A_{12}$ is the complex transmission impedance at the short circuit at the output, the parameter $A_{21}$ – the complex transmittance at no load at the output, the parameter $A_{22}$ is the complex transmission coefficient of the current in the short circuit at the output.

For mutual two-port ($Y_{12} = Y_{21}$) there are ratio

$$[A] = A_{11}A_{22} - A_{12}A_{21} = \frac{Y_{11}Y_{22}}{Y_{21}^2} - \frac{[Y]}{Y_{21}^2} = \frac{Y_{12}}{Y_{21}} = 1.$$

*B-parameters*. We choose the independent variables in expression (6.6) the current $\dot{I}_1$ and voltage $\dot{U}_1$ at the input of the two-port. We obtain the voltage $\dot{U}_2$ and current $\dot{I}_2$ at the output of the two-port in the expression (6.6):

$$\begin{cases} \dot{U}_2 = -\dfrac{Y_{11}}{Y_{12}}\dot{U}_1 + \dfrac{1}{Y_{12}}\dot{I}_1 = B_{11}\dot{U}_1 - B_{12}\dot{I}_1; \\ \dot{I}_2 = -\dfrac{[Y]}{Y_{12}}\dot{U}_1 + \dfrac{Y_{22}}{Y_{12}}\dot{I}_1 = B_{21}\dot{U}_1 - B_{22}\dot{I}_1. \end{cases} \qquad (6.28)$$

where

$$B_{11} = -\frac{Y_{11}}{Y_{12}};\ B_{12} = -\frac{1}{Y_{12}};\ B_{21} = -\frac{[Y]}{Y_{12}};\ B_{22} = -\frac{Y_{22}}{Y_{12}}. \qquad (6.29)$$

Elements $B_{11}$, $B_{12}$, $B_{21}$, $B_{22}$ are complex transfer functions from the output to the input and called $B$ - parameters of the two-port.

Expressions (6.29) can be rewritten in a matrix form:

$$\begin{bmatrix} \dot{U}_2 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ -\dot{I}_1 \end{bmatrix}.$$

From expression (6.28) it is clear that

$$B_{11} = \left.\frac{\dot{U}_2}{\dot{U}_1}\right|_{\dot{I}_1=0};\ B_{12} = -\left.\frac{\dot{U}_2}{\dot{I}_1}\right|_{\dot{U}_1=0};$$

$$B_{21} = \left.\frac{\dot{I}_2}{\dot{U}_1}\right|_{\dot{I}_1=0};\ B_{22} = -\left.\frac{\dot{I}_2}{\dot{I}_1}\right|_{\dot{U}_1=0}.$$

That is, $B$ - parameters can be determined experimentally by performing an without load at the input ($\dot{I}_1 = 0$) when calculating $B_{11}$, $B_{21}$ and testing a short circuit at the input ($\dot{U}_1 = 0$) when calculating $B_{12}$, $B_{22}$.

By physical nature, the parameter $B_{11}$ is a complex transmission coefficient of voltage an without load at the input, the parameter $B_{12}$ is the complex transmission impedance at the short circuit at the input, the parameter $B_{21}$ is the complex transmittance an without load at the input, the parameter $B_{22}$ is the complex transmission coefficient of the current at short circuiting at the input.

For mutual two-port ($Y_{12} = Y_{21}$) there is a ratio

$$[B] = B_{11}B_{22} - B_{12}B_{21} = \frac{Y_{11}Y_{22}}{Y_{12}^2} - \frac{[Y]}{Y_{21}^2} = \frac{Y_{12}}{Y_{21}} = 1.$$

With $A_{11} = B_{22}$, $A_{12} = B_{12}$, $A_{21} = B_{21}$, $A_{22} = B_{11}$. $A$- and $B$-parameters are called transmission parameters.

*H*-parameters. We select the independent variables in formula (6.12): the current $\dot{I}_1$ at the input and the voltage $\dot{U}_2$ at the output. We hold the voltage $\dot{U}_1$ at the input and the current $\dot{I}_2$ at the output of two-port in its expression (6.6):

$$\begin{cases} \dot{U}_1 = -\dfrac{1}{Y_{11}}\dot{I}_1 - \dfrac{Y_{12}}{Y_{11}}\dot{U}_2 = H_{11}\dot{I}_1 + H_{12}\dot{U}_2; \\ \dot{I}_2 = -\dfrac{Y_{21}}{Y_{11}}\dot{U}_1 + \dfrac{[Y]}{Y_{11}}\dot{U}_2 = H_{21}\dot{I}_1 + H_{22}\dot{U}_2. \end{cases} \qquad (6.30)$$

where

$$H_{11} = \frac{1}{Y_{11}};\ H_{12} = -\frac{Y_{12}}{Y_{11}};\ H_{21} = \frac{Y_{21}}{Y_{11}};\ H_{22} = \frac{[Y]}{Y_{11}}.$$

Elements $H_{11}$, $H_{12}$, $H_{21}$, $H_{22}$ are complex input, output and transfer functions of the two-port and are called *H*-parameters of the two-port.

Expression (6.30) can be rewritten in a matrix form

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{U}_2 \end{bmatrix}.$$

From expression (6.30) it is clear that

$$H_{11} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{U}_2=0};\ \ H_{12} = \left.\frac{\dot{U}_1}{\dot{U}_2}\right|_{\dot{I}_1=0};\ \ H_{21} = \left.\frac{\dot{I}_2}{\dot{I}_1}\right|_{\dot{U}_2=0};\ \ H_{22} = \left.\frac{\dot{I}_2}{\dot{U}_2}\right|_{\dot{I}_1=0}.$$

That is, *H*-parameters can be determined experimentally by performing the without load at the input ($\dot{I}_1 = 0$) when calculating $H_{12}$, $H_{22}$ and testing a short circuit at the output ($\dot{U}_2 = 0$) when calculating $H_{11}$, $H_{21}$.

By physical nature, the parameter $H_{11}$ is a complex input impedance at short circuit at the output, the parameter $H_{12}$ is a complex transmission coefficient of voltage without load at the input, the parameter $H_{21}$ is the complex transmission coefficient of the current in the short circuit at the output, the parameter $H_{22}$ is the complex output admittance at the without load at the input.

For mutual two-port ($Y_{12} = Y_{21}$) there is a ratio $H_{12} = -H_{21}$; *H*-parameters are widely for analyses transistors and transistor circuits.

**Example 6.3.**

In Fig. 6.3,b shows the scheme of replacement of the transistor through its physical parameters α, $r_e = r_1$, $r_b = r_2$, $r_k = r_3$. Find its *H*-parameters.

We write the system of loop equations for the circuit fig.6.3, b. Pre-convert the current source $\dot{I} = \alpha\dot{I}_1$ to the voltage source $\dot{E} = \dot{I}r_k = \alpha\dot{I}_1 r_k$:

$$\begin{cases} \dot{I}_1 r_e + (\dot{I}_1 + \dot{I}_2) r_b - \dot{U}_1 = 0; \\ -\dot{I}_2 r_k - (\dot{I}_1 + \dot{I}_2) r_b - \dot{E} + \dot{U}_2 = 0, \end{cases} \tag{6.31}$$

or

$$\begin{cases} (r_e + r_b)\dot{I}_1 + r_b \dot{I}_2 = \dot{U}_1; \\ (r_b + \alpha r_k)\dot{I}_1 + (r_b + r_k)\dot{I}_2 = \dot{U}_2. \end{cases}$$

Usually in the transistor $r_b \ll r_k$. Then

$$\begin{cases} (r_e + r_b)\dot{I}_1 + r_b \dot{I}_2 = \dot{U}_1; \\ \alpha r_k \dot{I}_1 + r_k \dot{I}_2 = \dot{U}_2. \end{cases} \tag{6.32}$$

Define $\dot{I}_2$ from the second equation (6.32):

$$\dot{I}_2 = \frac{1}{r_k}\dot{U}_2 - \alpha\dot{I}_1. \tag{6.33}$$

We substitute (6.33) in the first equation (6.32). Get the system

$$\begin{cases} \dot{U}_1 = [r_e + (1 - \alpha) r_b]\dot{I}_1 + \dfrac{r_b}{r_k}\dot{U}_2; \\ \dot{I}_2 = -\alpha\dot{I}_1 + \dfrac{1}{r_k}\dot{U}_2. \end{cases} \tag{6.34}$$

Comparing equations (6.34) and (6.30), we obtain

$$H_{11} = r_e + (1 - \alpha) r_b;\ H_{12} = \frac{r_b}{r_k};\ H_{21} = -\alpha;\ H_{22} = \frac{1}{r_k}.$$

*G-parameters*. We choose the independent variables in the expression (6.6) of the input voltage $\dot{U}_1$ and the output current $\dot{I}_2$. Find the current $\dot{I}_1$ at the input and the voltage $\dot{U}_2$ at the output of the two port:

$$\begin{cases} \dot{I}_1 = -\dfrac{[Y]}{Y_{22}}\dot{U}_1 + \dfrac{Y_{12}}{Y_{22}}\dot{I}_2 = G_{11}\dot{U}_1 - G_{12}\dot{I}_2; \\ \dot{U}_2 = -\dfrac{Y_{21}}{Y_{22}}\dot{U}_1 - \dfrac{1}{Y_{22}}\dot{I}_2 = G_{21}\dot{U}_1 - G_{22}\dot{I}_2, \end{cases} \tag{6.35}$$

where

$$G_{11} = \frac{[Y]}{Y_{22}};\ G_{12} = \frac{Y_{12}}{Y_{22}};\ G_{21} = -\frac{Y_{21}}{Y_{22}};\ G_{22} = \frac{1}{Y_{22}}.$$

Elements $G_{11}$, $G_{12}$, $G_{21}$, $G_{22}$ are complex input, output and transmitting functions of the two-port and are called the parameters of the two-port.

The expression (6.35) can be rewritten in a matrix form

$$\begin{bmatrix} \dot{I}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_1 \\ \dot{I}_2 \end{bmatrix}.$$

From expression (6.35) it is clear that

$$G_{11} = \left.\frac{\dot{I}_1}{\dot{U}_1}\right|_{\dot{I}_2=0}; \quad G_{12} = \left.\frac{\dot{I}_1}{\dot{I}_2}\right|_{\dot{U}_1=0}; \quad G_{21} = \left.\frac{\dot{U}_2}{\dot{U}_1}\right|_{\dot{I}_2=0}; \quad G_{22} = \left.\frac{\dot{U}_2}{\dot{I}_2}\right|_{\dot{U}_1=0}.$$

That is, $G$ - parameters can be determined experimentally, by performing a short circuit test at the input ($\dot{U}_1 = 0$) when calculating $G_{12}$ and $G_{22}$ and without load at the output ($\dot{I}_2 = 0$) when calculating $G_{11}$ and $G_{21}$.

By physical nature, the parameter $G_{11}$ is the complex input conductivity at without load at the output a, the parameter $G_{12}$ is the complex transmission coefficient of current at the short circuit at the input, the parameter $G_{21}$ is the complex transmission coefficient of the voltage at without load at the output, the parameter $G_{22}$ is the complex input impedance at the short circuit at the output.

For mutual two ports ($Y_{12} = Y_{21}$) there is a ratio $G_{12} = -G_{21}$;

*H*- and *G-parameters are called hybrid parameters*; *H*- and *G* - parameters of the two-port are dual.

All two-ports parameters are expressed in terms of *Y*-parameters. In the same way, any system of parameters can be expressed through another system.

**Example 6.4.**

Select *H*-parameters through *A*-parameters.

The system of two-ports equations in *A*-parameters using from (6.26):

$$\begin{cases} \dot{U}_1 = A_{11}\dot{U}_2 - A_{12}\dot{I}_2; \\ \dot{I}_1 = A_{21}\dot{U}_2 - A_{22}\dot{I}_2. \end{cases}$$

The system of two-port equations in *H*-parameters of expression using from (6.30)

$$\begin{cases} \dot{U}_1 = H_{11}\dot{I}_1 + H_{12}\dot{U}_2; \\ \dot{I}_2 = H_{21}\dot{I}_1 + H_{22}\dot{U}_2. \end{cases}$$

From the second equation (6.26) we get

$$\dot{I}_2 = -\frac{1}{A_{22}}\dot{I}_1 + \frac{A_{21}}{A_{22}}\dot{U}_2. \qquad (6.36)$$

We substitute (6.36) in the first equation (6.26), then

$$\dot{U}_1 = A_{11}\dot{U}_2 - A_{12}\left(-\frac{1}{A_{22}}\dot{I}_1 + \frac{A_{21}}{A_{22}}\dot{U}_2\right) =$$

$$= \frac{A_{12}}{A_{22}}\dot{I}_1 + \left(A_{11} - \frac{A_{12}A_{21}}{A_{22}}\right)\dot{U}_2 = \frac{A_{12}}{A_{22}}\dot{I}_1 + \frac{[A]}{A_{22}}\dot{U}_2, \qquad (6.37)$$

where

$$[A] = A_{11}A_{22} - A_{12}A_{21}.$$

That is, the expressions (6.36) and (6.37) can be written as a system of equations

$$\begin{cases} \dot{U}_1 = \dfrac{A_{12}}{A_{22}}\dot{I}_1 + \dfrac{[A]}{A_{22}}\dot{U}_2; \\ \dot{I}_2 = -\dfrac{1}{A_{22}}\dot{I}_1 + \dfrac{A_{21}}{A_{22}}\dot{U}_2. \end{cases} \qquad (6.38)$$

Comparing the systems (6.38) and (6.30), we obtain

$$H_{11} = \frac{A_{12}}{A_{22}};\, H_{12} = \frac{[A]}{A_{22}};\, H_{21} = -\frac{1}{A_{22}};\, H_{22} = \frac{A_{21}}{A_{22}}.$$

Table 6.1 shows the correlation of parameters for mutual symmetrice two-ports, and tables 6.2 and 6.3 - the formulas for the conversion of some parameters of two-ports through other.

Table 6.1

| Two-ports | Y | Z | A | B | H | G |
|---|---|---|---|---|---|---|
| Reciprocal | $Y_{12} = Y_{21}$ | $Z_{12} = Z_{21}$ | $[A] = 1$ | $[B] = 1$ | $H_{12} = -H_{21}$ | $G_{12} = -G_{21}$ |
| Symmetric | $Y_{11} = Y_{22}$ | $Z_{11} = Z_{22}$ | $A_{11} = A_{22}$ | $B_{11} = B_{22}$ | $[H] = 1$ | $[G] = 1$ |

Table 6.2

| | Y | Z | A |
|---|---|---|---|
| Y | $\begin{matrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{matrix}$ | $\begin{matrix} \dfrac{Z_{22}}{[Z]} & -\dfrac{Z_{12}}{[Z]} \\ -\dfrac{Z_{21}}{[Z]} & \dfrac{Z_{11}}{[Z]} \end{matrix}$ | $\begin{matrix} \dfrac{A_{22}}{A_{12}} & -\dfrac{[A]}{A_{12}} \\ -\dfrac{1}{A_{12}} & \dfrac{A_{11}}{A_{12}} \end{matrix}$ |

| | | | |
|---|---|---|---|
| Z | $\begin{matrix} \frac{Y_{22}}{[Y]} & -\frac{Y_{12}}{[Y]} \\ -\frac{Y_{21}}{[Y]} & \frac{Y_{11}}{[Y]} \end{matrix}$ | $\begin{matrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{matrix}$ | $\begin{matrix} \frac{A_{11}}{A_{21}} & \frac{[A]}{A_{21}} \\ \frac{1}{A_{21}} & \frac{A_{22}}{A_{21}} \end{matrix}$ |
| A | $\begin{matrix} -\frac{Y_{22}}{Y_{21}} & -\frac{1}{Y_{21}} \\ -\frac{[Y]}{Y_{21}} & -\frac{Y_{11}}{Y_{21}} \end{matrix}$ | $\begin{matrix} \frac{Z_{11}}{Z_{21}} & \frac{[Z]}{Z_{21}} \\ \frac{1}{Z_{21}} & \frac{Z_{22}}{Z_{21}} \end{matrix}$ | $\begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix}$ |
| B | $\begin{matrix} -\frac{Y_{11}}{Y_{12}} & -\frac{1}{Y_{12}} \\ -\frac{[Y]}{Y_{12}} & -\frac{Y_{22}}{Y_{12}} \end{matrix}$ | $\begin{matrix} \frac{Z_{22}}{Z_{12}} & \frac{[Z]}{Z_{12}} \\ \frac{1}{Z_{12}} & \frac{Z_{11}}{Z_{12}} \end{matrix}$ | $\begin{matrix} \frac{A_{22}}{[A]} & \frac{A_{12}}{[A]} \\ \frac{A_{21}}{[A]} & \frac{A_{11}}{[A]} \end{matrix}$ |
| H | $\begin{matrix} \frac{1}{Y_{11}} & -\frac{Y_{12}}{Y_{11}} \\ \frac{Y_{21}}{Y_{11}} & -\frac{[Y]}{Y_{11}} \end{matrix}$ | $\begin{matrix} \frac{[Z]}{Z_{22}} & \frac{Z_{12}}{Z_{22}} \\ -\frac{Z_{21}}{Z_{22}} & \frac{1}{Z_{22}} \end{matrix}$ | $\begin{matrix} \frac{A_{12}}{A_{22}} & \frac{[A]}{A_{22}} \\ -\frac{1}{A_{22}} & \frac{A_{21}}{A_{22}} \end{matrix}$ |
| G | $\begin{matrix} \frac{[Y]}{Y_{22}} & \frac{Y_{12}}{Y_{22}} \\ -\frac{Y_{21}}{Y_{22}} & \frac{1}{Y_{22}} \end{matrix}$ | $\begin{matrix} \frac{1}{Z_{11}} & -\frac{Z_{12}}{Z_{11}} \\ \frac{Z_{21}}{Z_{11}} & -\frac{[Z]}{Z_{11}} \end{matrix}$ | $\begin{matrix} \frac{A_{21}}{A_{11}} & -\frac{[A]}{A_{11}} \\ \frac{1}{A_{11}} & \frac{A_{12}}{A_{11}} \end{matrix}$ |
| | B | H | G |
| Y | $\begin{matrix} \frac{B_{11}}{B_{12}} & -\frac{1}{B_{12}} \\ -\frac{[B]}{B_{12}} & -\frac{B_{22}}{B_{12}} \end{matrix}$ | $\begin{matrix} \frac{1}{H_{11}} & -\frac{H_{12}}{H_{11}} \\ \frac{H_{21}}{H_{11}} & \frac{[H]}{H_{11}} \end{matrix}$ | $\begin{matrix} \frac{[G]}{G_{22}} & \frac{G_{12}}{G_{22}} \\ -\frac{G_{21}}{G_{22}} & \frac{1}{G_{22}} \end{matrix}$ |
| Z | $\begin{matrix} \frac{B_{22}}{B_{21}} & \frac{1}{B_{21}} \\ \frac{[B]}{B_{21}} & \frac{B_{11}}{B_{21}} \end{matrix}$ | $\begin{matrix} \frac{[H]}{H_{22}} & \frac{H_{12}}{H_{22}} \\ -\frac{H_{21}}{H_{22}} & \frac{1}{H_{22}} \end{matrix}$ | $\begin{matrix} \frac{1}{G_{11}} & -\frac{G_{12}}{G_{11}} \\ \frac{G_{21}}{G_{11}} & \frac{[G]}{G_{11}} \end{matrix}$ |

| | | | |
|---|---|---|---|
| A | $\begin{matrix} \frac{B_{22}}{[B]} & \frac{B_{12}}{[B]} \\ \frac{B_{21}}{[B]} & \frac{B_{11}}{[B]} \end{matrix}$ | $\begin{matrix} -\frac{[H]}{H_{21}} & -\frac{H_{11}}{H_{21}} \\ -\frac{H_{22}}{H_{21}} & -\frac{1}{H_{21}} \end{matrix}$ | $\begin{matrix} \frac{1}{G_{21}} & \frac{G_{22}}{G_{21}} \\ \frac{G_{11}}{G_{21}} & \frac{[G]}{G_{21}} \end{matrix}$ |
| B | $\begin{matrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{matrix}$ | $\begin{matrix} \frac{1}{H_{12}} & \frac{H_{11}}{H_{12}} \\ \frac{H_{22}}{H_{12}} & \frac{[H]}{H_{12}} \end{matrix}$ | $\begin{matrix} -\frac{[G]}{G_{12}} & -\frac{G_{22}}{G_{12}} \\ -\frac{G_{11}}{G_{12}} & -\frac{1}{G_{12}} \end{matrix}$ |
| H | $\begin{matrix} \frac{B_{12}}{B_{11}} & \frac{1}{B_{11}} \\ \frac{[B]}{B_{11}} & \frac{B_{21}}{B_{11}} \end{matrix}$ | $\begin{matrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{matrix}$ | $\begin{matrix} \frac{G_{22}}{[G]} & -\frac{G_{12}}{[G]} \\ -\frac{G_{21}}{[G]} & \frac{G_{11}}{[G]} \end{matrix}$ |
| G | $\begin{matrix} \frac{B_{21}}{B_{22}} & -\frac{1}{B_{22}} \\ \frac{[B]}{B_{22}} & \frac{B_{12}}{B_{22}} \end{matrix}$ | $\begin{matrix} \frac{H_{22}}{[H]} & -\frac{H_{12}}{[H]} \\ -\frac{H_{21}}{[H]} & \frac{H_{11}}{[H]} \end{matrix}$ | $\begin{matrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{matrix}$ |

*One-side dparameters*. For two-ports, which have large geometric sizes, for example, the communication lines, measuring voltages and currents at the input and output of the two-ports is not convenient in the experimental determination of its parameters. In this case,

One-sideds parameters are used, which represent a combination of immittance parameters *Y* and *Z*:

$$Z_{1x} = \frac{1}{Y_{1x}} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{I}_2=0} = Z_{11}; \tag{6.39}$$

$$Z_{2x} = \frac{1}{Y_{2x}} = \left.\frac{\dot{U}_2}{\dot{I}_2}\right|_{\dot{I}_1=0} = Z_{22}; \tag{6.40}$$

$$Z_{1k} = \frac{1}{Y_{1k}} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{U}_2=0} = \frac{1}{Y_{11}}; \tag{6.41}$$

$$Z_{2k} = \frac{1}{Y_{2k}} = \left.\frac{\dot{U}_2}{\dot{I}_2}\right|_{\dot{U}_1=0} = \frac{1}{Y_{22}}. \tag{6.42}$$

For symmetrical two-ports ($Z_{11} = Z_{22}, Y_{11} = Y_{22}$)

$$Z_{1x} = Z_{2x} = Z_x; Z_{1k} = Z_{2k} = Z_k.$$

Of the four parameters according to the relations (6.39) - (6.42), only three are independent, since there is a correlation

$$\frac{Z_{1x}}{Z_{1k}} = \frac{Z_{2x}}{Z_{2k}}.$$

In expressions (6.39) - (6.42) there aren't transfer functions. Therefore one-sided parameters are obtained only for two-ports description of reciprocal circuits.

*Y*-, *Z*-, *A*-, *B*-, *H*-, *G*-parameters and one-sided parameters are given for two-ports don't account influence of external circuit. Therefore, they are called primary parameters.

### 6.4.Equivalent circuits for replacing two-ports

With systems of two-ports equations in *Y*-, *Z*-, *A*-, *B*-, *H*-, *G*-parameters it is possible to construct equivalent circuits for replacing two-ports. So the system of equations (6.9) in *Y*-parameters can be put in correspondence with the equivalent scheme (fig. 6.6).

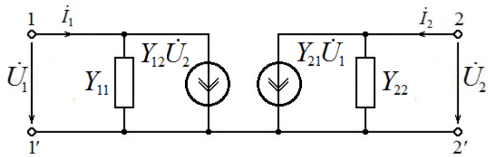



Fig. 6.6

Here, the sources of the current $Y_{12}\dot{U}_2$ corresponds to the second component in the right side (6.9). The first component are provided with conductors $Y_{11}$, $Y_{22}$ and voltages $\dot{U}_1$, $\dot{U}_2$.

The scheme of Fig. 6.7 also satisfies the system (6.9).

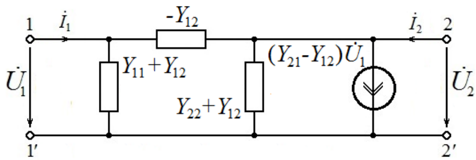



Fig.6.7

Really with a short circuit at the input ($\dot{U}_1 = 0$)

$$\dot{I}_1 = -\dot{U}_2 \cdot (-Y_{12}) = Y_{12}\dot{U}_2; \;\; Y_{12} = \left.\frac{\dot{I}_1}{\dot{U}_2}\right|_{\dot{U}_1=0};$$

$$\dot{I}_2 = \dot{U}_2 \cdot (-Y_{12} + Y_{22} + Y_{12}) = Y_{22}\dot{U}_2; \;\; Y_{22} = \left.\frac{\dot{I}_2}{\dot{U}_2}\right|_{\dot{U}_1=0.}.$$

With a short circuit at the input ($\dot{U}_2 = 0$)

$$\dot{I}_1 = \dot{U}_1 \cdot (Y_{11} + Y_{12} - Y_{12}) = Y_{11}\dot{U}_1; \;\; Y_{11} = \left.\frac{\dot{I}_1}{\dot{U}_1}\right|_{\dot{U}_2=0};$$

$$\dot{I}_2 = -\dot{U}_1 \cdot (-Y_{12}) + (Y_{21} - Y_{12})\dot{U}_1 = Y_{21}\dot{U}_1; \;\; Y_{21} = \left.\frac{\dot{I}_2}{\dot{U}_1}\right|_{\dot{U}_2=0}.$$

That is, the parameters $Y_{11}$, $Y_{12}$, $Y_{21}$, $Y_{22}$ obtained correspond to the relation (6.10), so the scheme of Fig. 6.7, satisfy the system (6.9).

For a mutual two-port ($Y_{12} = Y_{21}$), the circuit of the substitution is simplified (fig. 6.8), since the dependent current source $(Y_{21} - Y_{12})\dot{U}_1$ disappears.

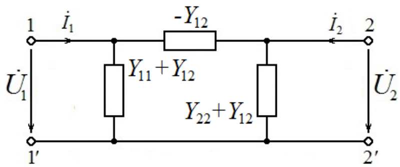



Fig. 6.8

The system of equations (6.23) in $Z$-parameters can be matched to the equivalent scheme (Fig. 6.9).

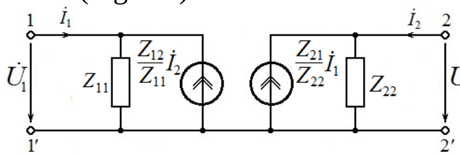



Fig. 6.9

Here, a current source $\frac{Z_{12}}{Z_{11}}\dot{I}_2$ with a parallel impedance $Z_{11}$ can be converted into a voltage source $\frac{Z_{12}}{Z_{11}}Z_{11}\dot{I}_2 = Z_{12}\dot{I}_2$ corresponding to the second term on the right side of equation (6.23), the first term in which is provided by current $\dot{I}_1$ and impedance $Z_{11}$. The current source $\frac{Z_{21}}{Z_{22}}\dot{I}_1$ with a parallel impedance $Z_{22}$ can be converted into a voltage source $\frac{Z_{21}}{Z_{22}}Z_{22}\dot{I}_1 = Z_{21}\dot{I}_1$, corresponding to the first term on the right side of equation (6.23), the second term being provided by current $\dot{I}_2$ and resistance $Z_{22}$.

The scheme of fig. 6.10 also satisfies the system (6.23).

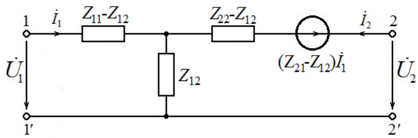



Fig. 6.10

Indeed, at idling at the input ($\dot{I}_1 = 0$)

$$\dot{U}_1 = \dot{I}_2 Z_{12};\quad Z_{12} = \left.\frac{\dot{U}_1}{\dot{I}_2}\right|_{\dot{I}_1=0};$$

$$\dot{U}_2 = \dot{I}_2 Z_{22};\quad Z_{22} = \left.\frac{\dot{U}_2}{\dot{I}_2}\right|_{\dot{I}_1=0}.$$

At idling at the output ($\dot{I}_2 = 0$)

$$\dot{I}_1(Z_{11} - Z_{12}) + \dot{I}_1 Z_{12} = \dot{U}_1;\quad Z_{11} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{I}_2=0};$$

$$\dot{U}_2 - \dot{I}_1 Z_{12} - (Z_{21} - Z_{12})\dot{I}_1 = 0;\quad Z_{21} = \left.\frac{\dot{U}_2}{\dot{I}_1}\right|_{\dot{I}_2=0}.$$

That is, the obtained parameters $Z_{11}$, $Z_{12}$, $Z_{21}$, $Z_{22}$ correspond to the relation (6.24), therefore, the scheme of Fig. 6.10 satisfies the system (6.23).

For a mutual two-port ($Z_{12} = Z_{21}$), the scheme of substitution is simplified (Fig. 6.11), since the dependent voltage source $(Z_{21} - Z_{12})$ is absent.

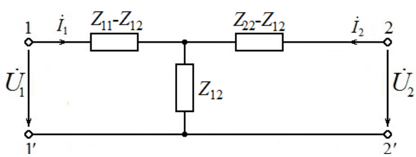



Fig. 6.11

The system of equations (6.30) in *H*-parameters can be placed in accordance with the equivalent scheme (Fig. 6.12)

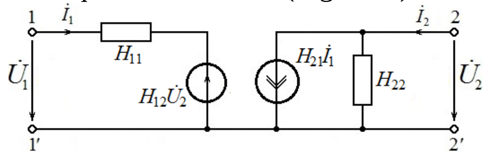



Fig. 6.12

Here, the voltage source $H_{12}\dot{U}_2$ corresponds to the second term on the right side of the first equation (6.30), the first term in which is provided by current $\dot{I}_1$ and impedance $H_{11}$. The current source $H_{21}\dot{I}_1$ corresponds to the first term in the right side of the second equation (6.30), the second term in which is provided by voltage $\dot{U}_2$ and conductivity $H_{22}$.

The system of equations (6.35) in *G*-parameters can be placed in accordance with the equivalent circuit of fig. 6.13. Here, the current source $G_{12}\dot{I}_2$ corresponds to the second part of the right-hand side of the equation (6.35), the first term of which is provided by voltage $\dot{U}_1$ and conductivity $G_{11}$. The voltage source $G_{21}\dot{U}_1$ corresponds to the first term of the right-hand side of the second equation (6.35), the second term in which is provided by current $\dot{I}_2$ and impedance $G_{22}$.

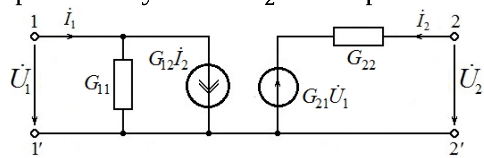



Fig. 6.13

Equivalent circuits for replacing two-port, built on the basis of *A*- and *B*-transmission parameters, are usually not used.

**6.5 Complex input and transfer functions of the two-port**

The primary ones are called *Y*-, *Z*-, *A*-, *B*-, *H*-, *G*- two-port parameters. They not depend on the outer circuit. Complex functions of the two-port – the ratio of the voltages and currents on its terminals –

taking into account the external circuit - are called secondary parameters of the two-port. Consider them.

*Input conductivity and input impedance*

$$Y_{in1} = Y_{in} = \frac{\dot{I}_1}{\dot{U}_1};\;\; Z_{in1} = Z_{in} = \frac{\dot{U}_1}{\dot{I}_1}. \tag{6.43}$$

*Output conductivity and output impedance*

$$Y_{in2} = Y_{out} = \frac{\dot{I}_2}{\dot{U}_2};\;\; Z_{in2} = Z_{out} = \frac{\dot{U}_2}{\dot{I}_2}.$$

For Fig. 6.4 of the equations of the two-ports in *Y*-parameters we have

$$\begin{cases} \dot{I}_1 = Y_{11}\dot{U}_1 + Y_{12}\dot{U}_2; \\ \dot{I}_2 = Y_{21}\dot{U}_1 + Y_{22}\dot{U}_2 = -Y_l\dot{U}_2. \end{cases} \tag{6.44}$$

where

$$Y_l = \frac{1}{Z_l}.$$

That is

$$Y_{21}\dot{U}_1 = -(Y_{22} + Y_l)\dot{U}_2,$$

where

$$\dot{U}_2 = -\frac{Y_{21}}{Y_{22} + Y_l}\dot{U}_1. \tag{6.45}$$

Now we can find from expressions (6.43 – 6.45)

$$Y_{in1} = Y_{in} = \frac{\dot{I}_1}{\dot{U}_1} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_l}.$$

On the principle of duality

$$Z_{in1} = Z_{in} = \frac{\dot{U}_1}{\dot{I}_1} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22} + Z_l}.$$

Similarly, from the output clamps (for $E_{in} = 0$)

$$Y_{in2} = Y_{out} = \frac{\dot{I}_2}{\dot{U}_2} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_{11} + Y_{inp}};$$

$$Z_{in2} = Z_{out} = \frac{\dot{U}_2}{\dot{I}_2} = Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11} + Z_{inp}},$$

where

$$Y_{inp} = \frac{1}{Z_{inp}}.$$

*Transmitting function or transmission factor by voltage*

$$\dot{K}_U = K_U e^{i\varphi_U} = \frac{\dot{U}_2}{\dot{U}_1}, \qquad (6.46)$$

where $K_U$, $\varphi_U$ - the module and the argument of the transfer function on the voltage.

For Fig.6.4 of the equations of the two-port in *A*-parameters (6.26), taking into account the expression (6.45), we have:

$$\dot{K}_U = \frac{\dot{U}_2}{\dot{U}_1} = -\frac{\dot{U}_2}{A_{11}\dot{U}_2 - A_{12}\dot{I}_2} = \frac{1}{A_{11} + A_{12}Y_l} = -\frac{Y_{21}}{Y_{22} + Y_l}. \quad (6.47)$$

*Transmitting function or transfer factor by current*

$$\dot{K}_I = K_I e^{i\varphi_I} = -\frac{\dot{I}_2}{\dot{I}_1}, \qquad (6.48)$$

where $K_I$, $\varphi_I$ – the module and the argument of the transfer function by current.

For fig. 6.4 of the equations of the two-port in *A*- parameters (6.26) we obtain the principle of duality

$$\dot{K}_I = -\frac{\dot{I}_2}{\dot{I}_1} = -\frac{\dot{I}_2}{A_{21}\dot{U}_2 - A_{22}\dot{I}_2} = \frac{1}{A_{21}Z_l + A_{22}} = \frac{Z_{21}}{Z_{22} + Z_l}. \quad (6.49)$$

*Operating voltage ratio*

$$\dot{K}_{U\,w} = \frac{\dot{U}_2}{\dot{E}_{in}}.$$

For Fig.6.4, taking into account the expression (6.47), we obtain

$$\dot{K}_{U\,w} = \frac{\dot{U}_2}{Z_{inp}Y_{in}\dot{U}_1 + \dot{U}_1} = \frac{\dot{K}_U}{1 + Z_{inp}Y_{in}} =$$

$$= \frac{-\dfrac{Y_{21}}{Y_{22} + Y_l}}{1 + \dfrac{Y_{in}}{Y_{inp}}} = -\frac{Y_{21}}{(Y_{22} + Y_l)\left(1 + \dfrac{Y_{in}}{Y_{inp}}\right)}.$$

*Transmitting conductivity and transmission impedance*

$$Y_{tr} = -\frac{\dot{I}_2}{\dot{U}_1}; \quad Z_{tr} = \frac{\dot{U}_2}{\dot{I}_1}.$$

For fig. 4, taking into account the expressions (6.46) and (6.48), we obtain

$$Y_{tr} = \frac{Y_l \dot{U}_2}{\dot{U}_1} = Y_l \dot{K}_U;$$

$$Z_{tr} = -\frac{\dot{I}_2 Z_l}{\dot{I}_1} = Z_l \dot{K}_I.$$

Table 6.3 is given expressions for calculation circuit complex function of two- ports through *Y*- and Z-parameters and using determinants of matrixes loop impedances (MLI) and matrixes of node conductivity (MNC).

Table 6.3

| Function | Y, Z | MLI | MNC |
|---|---|---|---|
| $Y_{in} = \frac{\dot{I}_1}{\dot{U}_1}$ | $Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_l}$ | $\frac{\Delta_{11} + Z_l \Delta_{11,22}}{\Delta + Z_l \Delta_{22}}$ | $\frac{\Delta + Y_l \Delta_{22}}{\Delta_{11} + Y_l \Delta_{11,22}}$ |
| $Z_{in} = \frac{\dot{U}_1}{\dot{I}_1}$ | $Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22} + Z_l}$ | $\frac{\Delta + Z_l \Delta_{22}}{\Delta_{11} + Z_l \Delta_{11,22}}$ | $\frac{\Delta_{11} + Y_l \Delta_{11,22}}{\Delta + Y_l \Delta_{22}}$ |
| $Y_{out} = \frac{\dot{I}_2}{\dot{U}_2}$ | $Y_{22} - \frac{Y_{12}Y_{21}}{Y_{11} + Y_{inp}}$ | $\frac{\Delta_{22} + Z_{inp} \Delta_{11,22}}{\Delta + Z_{inp} \Delta_{11}}$ | $\frac{\Delta + Y_{inp} \Delta_{11}}{\Delta_{22} + Y_{inp} \Delta_{11,22}}$ |
| $Z_{out} = \frac{\dot{U}_2}{\dot{I}_2}$ | $Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11} + Z_{inp}}$ | $\frac{\Delta + Z_{inp} \Delta_{11}}{\Delta_{22} + Z_{inp} \Delta_{11,22}}$ | $\frac{\Delta_{22} + Y_{inp} \Delta_{11,22}}{\Delta + Y_{inp} \Delta_{11}}$ |
| $\dot{K}_U = \frac{\dot{U}_2}{\dot{U}_1}$ | $-\frac{Y_{21}}{Y_{22} + Y_l}$ | $-\frac{Z_l \Delta_{12}}{\Delta + Z_l \Delta_{22}}$ | $\frac{\Delta_{12}}{\Delta_{11} + Y_l \Delta_{11,22}}$ |
| $\dot{K}_I = -\frac{\dot{I}_2}{\dot{I}_1}$ | $\frac{Z_{21}}{Z_{22} + Z_l}$ | $\frac{\Delta_{12}}{\Delta_{11} + Z_l \Delta_{11,22}}$ | $\frac{Y_l \Delta_{12}}{\Delta + Y_l \Delta_{22}}$ |
| $Y_{tr} = -\frac{\dot{I}_2}{\dot{U}_1}$ | $-\frac{Y_l Y_{21}}{Y_{22} + Y_l}$ | $\frac{\Delta_{12}}{\Delta + Z_l \Delta_{22}}$ | $\frac{Y_l \Delta_{12}}{\Delta_{11} + Y_l \Delta_{11,22}}$ |
| $Z_{tr} = \frac{\dot{U}_2}{\dot{I}_1}$ | $\frac{Z_l Z_{21}}{Z_{22} + Z_l}$ | $\frac{Z_l \Delta_{12}}{\Delta_{11} + Z_l \Delta_{11,22}}$ | $\frac{\Delta_{12}}{\Delta + Y_l \Delta_{22}}$ |

## 6.6. Characteristic parameters of the two-port

Characteristic parameters are convenient for the description of such types of two-ports as lines of communication, lines of delay, filters and others. They are used to describe the mutual two-ports. There are two types of characteristic parameters: characteristic impedance and characteristic transmission coefficient.

The characteristic impedances$Z_{c1}$, $Z_{c2}$ are the impedances between the terminals 1-1’ and 2-2’ respectively (see Fig. 6.4), in which the conditions of the matching are in place:

$$Z_{in} = Z_{inp} = Z_{11} - \frac{Z_{12}Z_{21}}{Z_{22} + Z_l}; \tag{6.50}$$

$$Z_{out} = Z_l = Z_{22} - \frac{Z_{12}Z_{21}}{Z_{11} + Z_{inp}}. \tag{6.51}$$

In this case there is no disturbance of the signal when it passes through the two-port and in the load the maximum power is transmitted.

Then

$$\begin{cases} Z_{c1} = Z_{in} = Z_{inp}; \\ Z_{c2} = Z_{out} = Z_l. \end{cases} \tag{6.52}$$

Substituting expressions (6.50) and (6.51) in (6.52) and taking into account the correlation of the table 6.1, we get

$$Z_{c1} = \sqrt{Z_{1x}Z_{1k}}; \;\; Z_{c2} = \sqrt{\frac{Z_{22}}{Y_{22}}}, \tag{6.53}$$

or through one-sided parameters (6.39) - (6.42) we have

$$Z_{c1} = \sqrt{Z_{1x}Z_{1k}}; \;\; Z_{c2} = \sqrt{Z_{2x}Z_{2k}}. \tag{6.54}$$

For symmetrical two-ports ($Z_{1x} = Z_{2x} = Z_x$, $Z_{1k} = Z_{2k} = Z_k$)

$$Z_{c1} = Z_{c2} = Z_c = \sqrt{Z_x Z_k}.$$

Characteristic transmission ratio γ is determined from the ratio

$$\dot{K}_U \dot{K}_I = \frac{\dot{U}_2}{\dot{U}_1}\left(-\frac{\dot{I}_2}{\dot{I}_1}\right) = e^{-2\gamma}. \tag{6.55}$$

Substituting expressions 6.47) and (6.49) into (6.55), we find

$$\gamma = \frac{1}{2}\ln\left[-\frac{(Y_{22} + Y_l)(Z_{22} + Z_l)}{Y_{21}Z_{21}}\right].$$

If in the load matching, in accordance with the expressions (6.52) and (6.53)

$$(Y_{22} + Y_l)(Z_{22} + Z_l) = \left(Y_{22} + \sqrt{\frac{Y_{22}}{Z_{22}}}\right)\left(Z_{22} + \sqrt{\frac{Z_{22}}{Y_{22}}}\right) =$$
$$= \left(\sqrt{Z_{22}Y_{22}} + 1\right)^2$$

and from tabl. 5.1

$$-Y_{21}Z_{21}=\frac{Y_{21}^2}{[Y]}=\frac{Y_{11}Y_{22}-[Y]}{[Y]}=\frac{Y_{11}Y_{22}}{[Y]}-1=$$
$$=Z_{22}Y_{22}-1=Z_{11}Y_{11}-1.$$

Then

$$\gamma=\frac{1}{2}\ln\frac{\sqrt{Z_{11}Y_{11}}+1}{\sqrt{Z_{11}Y_{11}}-1}=\frac{1}{2}\ln\frac{\sqrt{Z_{22}Y_{22}}+1}{\sqrt{Z_{22}Y_{22}}-1}. \quad (6.56)$$

By the formulas (6.39) - (6.42)

$$Z_{11}Y_{11}=\frac{Z_{1x}}{Z_{1k}};\quad Z_{22}Y_{22}=\frac{Z_{2x}}{Z_{2k}}.$$

Then from the expression (6.56)

$$\gamma=\frac{1}{2}\ln\frac{1+\sqrt{\frac{Z_{1k}}{Z_{1x}}}}{1-\sqrt{\frac{Z_{1k}}{Z_{1x}}}}=\frac{1}{2}\ln\frac{1+\sqrt{\frac{Z_{2k}}{Z_{2x}}}}{1-\sqrt{\frac{Z_{2k}}{Z_{2x}}}}. \quad (6.57)$$

By analogy with the ratio

$$\operatorname{arcth} y=\frac{1}{2}\ln\frac{y+1}{y-1}.$$

From formula (6.56) we find an expression γ due to hyperbolic functions

$$\gamma=\operatorname{arcth}\sqrt{Z_{11}Y_{11}}=\operatorname{arcth}\sqrt{Z_{22}Y_{22}}\,. \quad (6.58)$$

In general, the characteristic transmission coefficient is complex function

$$\gamma=\alpha+j\beta, \quad (6.59)$$

where α – the actual attenuation of the two port, β – the phase coefficient.

To find out the physical meaning of the values α and β, we will express $K_U$ and $K_I$ through α, β, γ. In the load matching.

$$\dot{U}_1=\dot{I}_1Z_{inp}=\dot{I}_1Z_{c1};\dot{U}_2=-\dot{I}_2Z_l=-\dot{I}_2Z_{c2}.$$

Then, by the formula (6.55)

$$e^{-2\gamma}=\frac{\dot{U}_2}{\dot{U}_1}\left(-\frac{\dot{I}_2}{\dot{I}_1}\right)=\frac{\dot{U}_2}{\dot{U}_1}\frac{\dot{U}_2}{Z_{c2}}\frac{Z_{c1}}{\dot{U}_1}=\frac{\dot{U}_2^2}{\dot{U}_1^2}\frac{Z_{c1}}{Z_{c2}}=K_U^2\frac{Z_{c1}}{Z_{c2}},$$

where

$$K_U=\sqrt{\frac{Z_{c2}}{Z_{c1}}}e^{-\gamma}. \quad (6.60)$$

Similarly

$$K_I = \sqrt{\frac{Z_{c1}}{Z_{c2}}} e^{-\gamma}.$$

From expression (6.60) can be written

$$\dot{K}_U = \sqrt{\frac{Z_{c2}}{Z_{c1}}} e^{j\frac{1}{2}(\varphi_{c2}-\varphi_{c2})} e^{-(\alpha+j\beta)} = \sqrt{\frac{Z_{c2}}{Z_{c1}}} e^{-\alpha} e^{j\left(\beta+\frac{\varphi_{c2}-\varphi_{c2}}{2}\right)}. \quad (6.61)$$

Similarly (6.61) you can write the expression $K_I$.

For a symmetrical two-port

$$Z_{c1} = Z_{c2},$$

so

$$K_U = K_I = e^{-\gamma} = e^{-(\alpha+j\beta)}.$$

From here

$$\alpha = \ln\frac{\dot{U}_1}{\dot{U}_2} = \ln\frac{\dot{I}_1}{\dot{I}_2} = \frac{1}{2}\ln\frac{S_1}{S_2}$$

where $S_1 = \dot{U}_1\dot{I}_1$, $S_2 = \dot{U}_2\dot{I}_2$ – the total powers at the input and output of the two-port.

Thus, the attenuation coefficient α determines the ratio of the voltage or current amplitudes at the input and output of the two-port, and the phase coefficient β indicates phase shift between the voltages or currents at the input and output of the two-port.

The unit of attenuation α is Neper (Np). For $\frac{\dot{U}_1}{\dot{U}_2} = e$, $\ln e = 1$Np, that is 1Np is the attenuation at which the amplitude of the voltage or current when the signal passes through two-port decreases in *e* times

The unit of the coefficient α may be Bell (B). For example $\frac{\dot{U}_1}{\dot{U}_2} = 10\,\frac{1}{2}ln10 = 1$, that 1B is attenuation, in which the signal power, when passing through the two-port decreases by 10 times.

Note: 1Np = 0.869B = 8.69dB; 1 decibel (dB) 10 is less times than 1B; 1dB = 0.1B = 0.115Np.

## 6.7. The equations of the two-ports in hyperbolic functions

Let's express *A*-parameters of the two-port through of its characteristic parameters. From expressions (6.53), (6.58) and tabl. 6.1 we get:

$$Z_{c1} = \sqrt{\frac{Z_{11}}{Y_{11}}} = \sqrt{\frac{A_{11}A_{12}}{A_{21}A_{22}}}; \quad Z_{c2} = \sqrt{\frac{Z_{22}}{Y_{22}}} = \sqrt{\frac{A_{22}A_{12}}{A_{21}A_{11}}}; \qquad (6.62)$$

$$\gamma = \operatorname{arcth}\sqrt{Z_{11}Y_{11}} = \operatorname{arcth}\sqrt{\frac{A_{11}A_{22}}{A_{21}A_{12}}},$$

that is

$$\operatorname{cth}\gamma = \sqrt{\frac{A_{11}A_{22}}{A_{21}A_{12}}}. \qquad (6.63)$$

From the expressions (6.62) and (6.63) we find

$$\begin{cases} A_{11} = \sqrt{\dfrac{Z_{c1}}{Z_{c2}}}\operatorname{ch}\gamma; \; A_{12} = \sqrt{Z_{c1}Z_{c2}}\operatorname{sh}\gamma; \\ A_{21} = \dfrac{1}{\sqrt{Z_{c1}Z_{c2}}}\operatorname{sh}\gamma; \; A_{11} = \sqrt{\dfrac{Z_{c2}}{Z_{c1}}}\operatorname{ch}\gamma; \end{cases} \qquad (6.64)$$

Thus, according to formulas (6.26) and (6.64) we write the system of their equations in hyperbolic functions:

$$\begin{cases} \dot{U}_1 = \sqrt{\dfrac{Z_{c1}}{Z_{c2}}}\left(\dot{U}_2\operatorname{ch}\gamma + Z_{c2}\dot{I}_2\operatorname{sh}\gamma\right); \\ \dot{I}_1 = \sqrt{\dfrac{Z_{c2}}{Z_{c1}}}\left(\dfrac{\dot{U}_2}{Z_{c2}}\operatorname{sh}\gamma + \dot{I}_2\operatorname{ch}\gamma\right). \end{cases}$$

## 6.8. The simplest two-ports

*Ideal transformer* (Fig. 6.14). This is a two-port, which does not dissipate and does not accumulate energy, that is, the ideal transformer is a passive device.

Powers at the input and output of the transformer are the same:

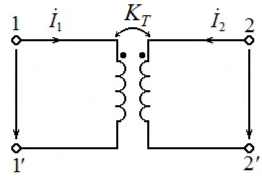



Fig. 6.14

$$u_1 i_1 = -u_2 i_2.$$

Transformation factor

$$n = \frac{u_1}{u_2} = -\frac{i_2}{i_1},$$

so

$$\begin{cases} u_1 = n u_2; \\ i_1 = -\dfrac{1}{n} i_2; \end{cases}$$

or

$$\begin{bmatrix} \dot{U}_1 \\ \dot{I}_1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \dfrac{1}{n} \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ -\dot{I}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \dot{U}_2 \\ -\dot{I}_2 \end{bmatrix}.$$

That is, the equation of an ideal transformer is expressed in *A*-parameters.

If

$$Y_{11} = \left.\frac{\dot{I}_1}{\dot{U}_1}\right|_{\dot{U}_2=0} \to \infty; \qquad Z_{11} = \left.\frac{\dot{U}_1}{\dot{I}_1}\right|_{\dot{I}_2=0} \to \infty,$$

then the ideal transformer does not have *Y*- and *Z*-the parameters.

*Gyrator*. This is a device that in most cases has a theoretical interest. Almost a device with properties of a gyrator can be constructed on microwave elements (MWE), as well as by means of transistors.

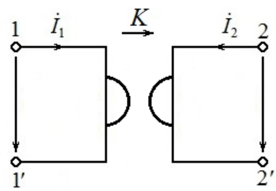



Fig. 6.15

The gyrator (Fig. 6.15) is described by the ratio

$$-\frac{u_1}{i_2} = \frac{u_2}{i_1} = k, \qquad (6.66)$$

where $k$ – the coefficient of gyration.

From expression (6.66)

$$\begin{cases} u_1 = -ki_2; \\ u_2 = ki_1; \end{cases} \qquad (6.67)$$

or

$$\begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \end{bmatrix} = \begin{bmatrix} 0 & -k \\ k & 0 \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \dot{I}_1 \\ \dot{I}_2 \end{bmatrix}. \qquad (6.68)$$

That is, the equation of the gyrator is expressed in the *Z*-parameters.

In the matrix *Z*-parameters of the girator (6.68) $Z_{12} \neq Z_{21}$, that is, the girator does not satisfy the principle of reciprocity.

If the powers at the input and output of the gyrator are the same

$$u_1 i_1 = -u_2 i_2,$$

then the gyrator is a passive element.

From expression (6.66), if $Z_l = -\frac{u_2}{i_2}$, then

$$Z_{in1} = \frac{\dot{U}_1}{\dot{I}_1} = -\frac{k\dot{I}_2}{\frac{\dot{U}_2}{k}} = \frac{k^2}{Z_l}$$

Thus, the gyrator converts the impedance $Z_l$ at the output impedance to the input impedance $\frac{k^2}{Z_l}$. At $k = 1$ the impedance is inverted into conductivity. For example, if, $Z_l = j\omega L$ tihen $\frac{1}{Z_l} = \frac{1}{j\omega L} = \frac{1}{j\omega C_e}$, ie $C_e = L$ (inductance 1H at the output is transformed into capacitance 1F at the input.)

The gyrator in the theory of circuit was introduced in 1948 by Telegen.

*Negative Convertor (KNI).* This is a device that allows you to convert any impedance of $Z_l$ into impedance $-Z_l$ with opposite sign. The negative impedance converter is described by the ratio

$$\frac{u_1}{i_1} = -\left(\frac{u_2}{-i_2}\right). \qquad (6.69)$$

Condition (6.69) is satisfied with

$$u_1 = ku_2; \quad i_1 = -k(-i_2), \qquad (6.70)$$

or

$$\begin{bmatrix}\dot{U}_1\\ \dot{I}_1\end{bmatrix}=\begin{bmatrix}k & 0\\ 0 & -k\end{bmatrix}\begin{bmatrix}\dot{U}_2\\ -\dot{I}_2\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\dot{U}_2\\ -\dot{I}_2\end{bmatrix}. \qquad (6.71)$$

In terms (6.70) and (6.71), the coefficient $k$ is simultaneously the transformation factor of voltage and current, that is, KNI can transform power.

### 6.9 Complex two-ports

Two-ports are called complicated, which can be represented as a combination of several simple two-ports. If known parameters of simple two-ports, then you can express complex parameters. There are several ways of connecting simple two-ports in the formation of complex two-port.

*Series connection* (fig. 6.16).

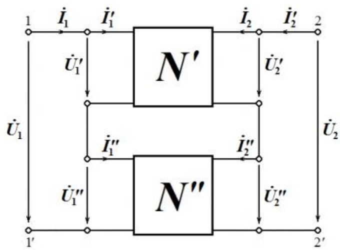



Fig. 6.16

With

$$\begin{bmatrix}\dot{U}_1\\ \dot{U}_2\end{bmatrix}=\begin{bmatrix}\dot{U}_1'\\ \dot{U}_2'\end{bmatrix}+\begin{bmatrix}\dot{U}_1''\\ \dot{U}_2''\end{bmatrix}=\begin{bmatrix}\dot{U}_1'+\dot{U}_1''\\ \dot{U}_2'+\dot{U}_2''\end{bmatrix},\ \begin{bmatrix}\dot{I}_1\\ \dot{I}_2\end{bmatrix}=\begin{bmatrix}\dot{I}_1'\\ \dot{I}_2'\end{bmatrix}=\begin{bmatrix}\dot{I}_1''\\ \dot{I}_2''\end{bmatrix},$$

then

$$\begin{bmatrix}\dot{U}_1\\ \dot{U}_2\end{bmatrix}=\begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix}\begin{bmatrix}\dot{I}_1\\ \dot{I}_2\end{bmatrix}=\left\{\begin{bmatrix}Z_{11}' & Z_{12}'\\ Z_{21}' & Z_{22}'\end{bmatrix}+\begin{bmatrix}Z_{11}'' & Z_{12}''\\ Z_{21}'' & Z_{22}''\end{bmatrix}\right\}\begin{bmatrix}\dot{I}_1\\ \dot{I}_2\end{bmatrix}.$$

That is, with the series connection of the two-ports their matrices of $Z$-parameters are summed up:

$$[Z]=[Z']+[Z''].$$

*Parallel connection* (fig. 6.17)

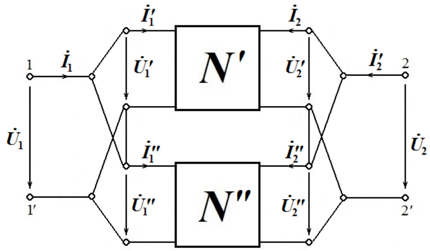



Fig. 6.17

With

$$\begin{bmatrix}\dot U_1\\ \dot U_2\end{bmatrix}=\begin{bmatrix}\dot U_1'\\ \dot U_2'\end{bmatrix}=\begin{bmatrix}\dot U_1''\\ \dot U_2''\end{bmatrix},\ \begin{bmatrix}\dot I_1\\ \dot I_2\end{bmatrix}=\begin{bmatrix}\dot I_1'\\ \dot I_2'\end{bmatrix}+\begin{bmatrix}\dot I_1''\\ \dot I_2''\end{bmatrix},$$

then

$$\begin{bmatrix}\dot I_1\\ \dot I_2\end{bmatrix}=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}\begin{bmatrix}\dot U_1\\ \dot U_2\end{bmatrix}=\left\{\begin{bmatrix}Y_{11}' & Y_{12}'\\ Y_{21}' & Y_{22}'\end{bmatrix}+\begin{bmatrix}Y_{11}'' & Y_{12}''\\ Y_{21}'' & Y_{22}''\end{bmatrix}\right\}\begin{bmatrix}\dot U_1\\ \dot U_2\end{bmatrix}.$$

that is, at the parallel connection of the two-ports their matrices of $Y$-parameters are summed up:

$$[Y]=[Y']+[Y''].$$

*Series-parallel connection* (fig. 6.18)

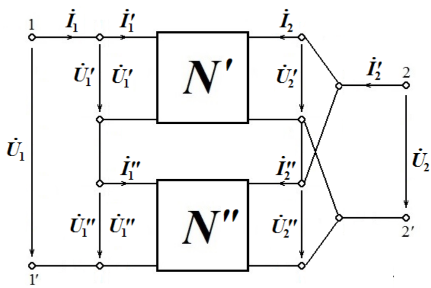



Fig. 6.18

With

$$\begin{bmatrix}\dot U_1\\ \dot I_2\end{bmatrix}=\begin{bmatrix}\dot U_1'\\ \dot I_2'\end{bmatrix}+\begin{bmatrix}\dot U_1''\\ \dot I_2''\end{bmatrix}=\begin{bmatrix}\dot U_1'+\dot U_1''\\ \dot I_2'+\dot I_2''\end{bmatrix},\ \begin{bmatrix}\dot I_1\\ \dot U_2\end{bmatrix}=\begin{bmatrix}\dot I_1'\\ \dot U_2'\end{bmatrix}=\begin{bmatrix}\dot I_1''\\ \dot U_2''\end{bmatrix},$$

then

$$\begin{bmatrix}\dot{U}_1\\ \dot{I}_2\end{bmatrix}=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\dot{I}_1\\ \dot{U}_2\end{bmatrix}=\left\{\begin{bmatrix}H'_{11} & H'_{12}\\ H'_{21} & H'_{22}\end{bmatrix}+\begin{bmatrix}H''_{11} & H''_{12}\\ H''_{21} & H''_{22}\end{bmatrix}\right\}\begin{bmatrix}\dot{I}_1\\ \dot{U}_2\end{bmatrix}.$$

That is, in the case of a series-parallel connection of the two-ports, their matrices of *Y*- parameters are summed up:

$$[H]=[H']+[H''].$$

*Parallel-series connection* (fig. 6.19).

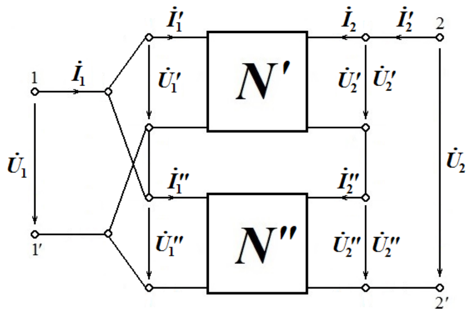



Fig. 6.19

With

$$\begin{bmatrix}\dot{I}_1\\ \dot{U}_2\end{bmatrix}=\begin{bmatrix}\dot{I}'_1\\ \dot{U}'_2\end{bmatrix}+\begin{bmatrix}\dot{I}''_1\\ \dot{U}''_2\end{bmatrix}=\begin{bmatrix}\dot{I}'_1+\dot{I}''_1\\ \dot{U}'_2+\dot{U}''_2\end{bmatrix},\ \begin{bmatrix}\dot{U}_1\\ \dot{I}_2\end{bmatrix}=\begin{bmatrix}\dot{U}'_1\\ \dot{I}'_2\end{bmatrix}=\begin{bmatrix}\dot{U}''_1\\ \dot{I}''_2\end{bmatrix},$$

then

$$\begin{bmatrix}\dot{I}_1\\ \dot{U}_2\end{bmatrix}=\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix}\dot{U}_1\\ \dot{I}_2\end{bmatrix}=\left\{\begin{bmatrix}G'_{11} & G'_{12}\\ G'_{21} & G'_{22}\end{bmatrix}+\begin{bmatrix}G''_{11} & G''_{12}\\ G''_{21} & G''_{22}\end{bmatrix}\right\}\begin{bmatrix}\dot{U}_1\\ \dot{I}_2\end{bmatrix}.$$

That is, at the parallel-serial connection of the two-port, their matrices of *G*- parameters are summed up:

$$[G]=[G']+[G''].$$

Cascade connection (fig. 6.20).

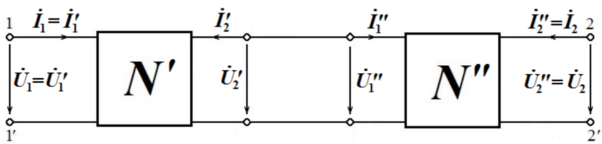



Fig. 6.20

With

$$\begin{bmatrix}\dot{U}_1\\ \dot{I}_1\end{bmatrix}=\begin{bmatrix}\dot{U}'_1\\ \dot{I}'_1\end{bmatrix};\ \begin{bmatrix}\dot{U}'_2\\ -\dot{I}'_2\end{bmatrix}=\begin{bmatrix}\dot{U}''_1\\ \dot{I}_1\end{bmatrix};\ \begin{bmatrix}\dot{U}_2\\ -\dot{I}''_2\end{bmatrix}=\begin{bmatrix}\dot{U}_2\\ -\dot{I}_2\end{bmatrix},$$

then

$$\begin{bmatrix}\dot{U}_1\\ \dot{I}_1\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}\dot{U}_2\\ -\dot{I}_2\end{bmatrix}=\begin{bmatrix}A'_{11} & A'_{12}\\ A'_{21} & A'_{22}\end{bmatrix}\begin{bmatrix}A''_{11} & A''_{12}\\ A''_{21} & A''_{22}\end{bmatrix}\begin{bmatrix}\dot{U}_2\\ -\dot{I}_2\end{bmatrix}$$

That is, at the cascade connection of the two-ports, their matrices of *A*-parameters are multiplied in the order of the location of the two-ports:

$$[A]=[A'][A''].$$

Of great practical importance is the cascade connection *n*-th number of two-ports with characteristic transmission coefficients $\dot{\gamma}_1$, $\dot{\gamma}_2$,…, $\dot{\gamma}_n$ and characteristic impedances $Z_{c1}$ and $Z_{c2}$, $Z_{c2}$ and $Z_{c3}$, …, $Z_{cn}$ and $Z_{cn+1}$ (fig.6.21).

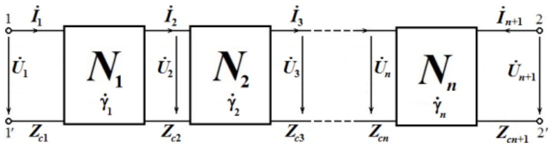



Fig. 6.21

The characteristic impedances of two-ports are matched, that is, the load $Z_l$ is matched with the output characteristic impedance $Z_{cn-1}$ of the *n*-th two-port. Its input impedance is equal to the characteristic impedance $Z_{cn}$ and is matched witch load of *n*-1 two-port etc. Input impedance of the first two-port also equal characteristic impedance $Z_{c1}$.

For the fig. 6.21 according to formula (6.60)

$$\dot{K}_U=\frac{\dot{U}_{n+1}}{\dot{U}_1}=\sqrt{\frac{Z_{cn+1}}{Z_{c1}}}e^{\dot{\gamma}},$$

where, $\dot{\gamma}=\dot{\gamma}_1+\ \dot{\gamma}_2+\cdots+\ \dot{\gamma}_n$, that is, the cascade connection of two-ports is equivalent to single two-port, whose characteristic impedances are equal to the input characteristic impedance of the first and the output characteristic impedance of the last two-port. The characteristic transfer coefficient of the resulting two-port is equal to the algebraic sum of the characteristic coefficients of transmission of the individual two-ports.

**Example 6.5**.

Find $A$-parameters of the two-port according to the circuit in Fig. 6.22.

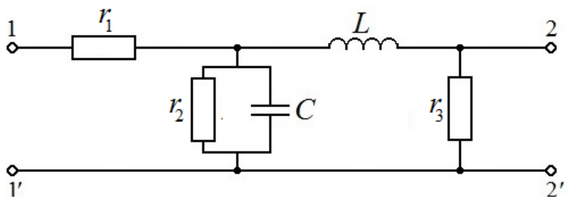



Fig. 6.22

Let us introduce the circuit in the form of a cascade connection of simple two-ports I-IV (Fig. 6.23); $A$-parameters for each of them are obtained by the formula (6.27):

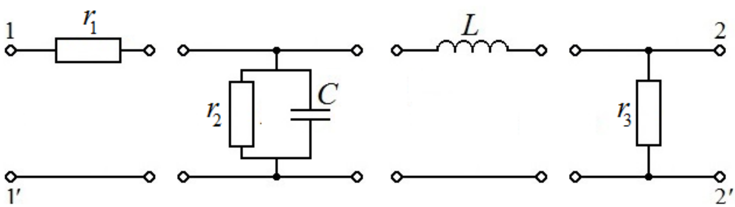



Fig. 6.23

$$[A^{\mathrm{I}}] = \begin{bmatrix} A_{11}^{\mathrm{I}} & A_{12}^{\mathrm{I}} \\ A_{21}^{\mathrm{I}} & A_{22}^{\mathrm{I}} \end{bmatrix} = \begin{bmatrix} 1 & r_1 \\ 0 & 1 \end{bmatrix};\ [A^{\mathrm{II}}] = \begin{bmatrix} A_{11}^{\mathrm{II}} & A_{12}^{\mathrm{II}} \\ A_{21}^{\mathrm{II}} & A_{22}^{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{r_2} + j\omega C & 1 \end{bmatrix};$$

$$[A^{\mathrm{III}}] = \begin{bmatrix} A_{11}^{\mathrm{III}} & A_{12}^{\mathrm{III}} \\ A_{21}^{\mathrm{III}} & A_{22}^{\mathrm{III}} \end{bmatrix} = \begin{bmatrix} 1 & j\omega L \\ 0 & 1 \end{bmatrix};\ [A^{\mathrm{IV}}] = \begin{bmatrix} A_{11}^{\mathrm{IV}} & A_{12}^{\mathrm{IV}} \\ A_{21}^{\mathrm{IV}} & A_{22}^{\mathrm{IV}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{r_3} & 1 \end{bmatrix};$$

Now

$$[A] = [A^{\mathrm{I}}][A^{\mathrm{II}}][A^{\mathrm{III}}][A^{\mathrm{IV}}] = \begin{bmatrix} 1 & r_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{r_2} + j\omega C & 1 \end{bmatrix} \begin{bmatrix} 1 & j\omega L \\ 0 & 1 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 1 & 0 \\ \frac{1}{r_3} & 1 \end{bmatrix} = \begin{bmatrix} 1 + r_1\left(\frac{1}{r_2} + j\omega C\right) & r_1 \\ \frac{1}{r_2} + j\omega C & 1 \end{bmatrix} \begin{bmatrix} 1 + \frac{j\omega L}{r_3} & j\omega L \\ \frac{1}{r_3} & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \left[1 + r_1\left(\frac{1}{r_2} + j\omega C\right)\right]\left(1 + \frac{j\omega L}{r_3}\right) + \frac{r_1}{r_3} & \left[1 + r_1\left(\frac{1}{r_2} + j\omega C\right)\right] j\omega L + r_1 \\ \left(\frac{1}{r_2} + j\omega C\right)\left(1 + \frac{j\omega L}{r_3}\right) + \frac{1}{r_3} & \left(\frac{1}{r_2} + j\omega C\right) j\omega L + 1 \end{bmatrix}$$

## Methodic instruction

By study of section “Bases of two-ports theory” it’s necessary to concentrate the base attention on system of two-ports parameters, note difference between primary and secondary parameters, as their questions are introductory to characteristically parameters, which are studied late. Study of characteristically parameters it’s necessary by explain the such kind of two-ports as filters and long line. Useful to acquaintance which simple two-ports and methods their connection in complex two-ports.

Literature [1 - 4], [14 - 16]

## Questions for self checking

1. What are the two-ports and how they are classified?
2. Write down the two-port equation in *Y*-, *Z*-, *A*-, *B*-, *H*-, *G*-parameters.
3. How to determine the parameters of the two-port experimentally?
4. What are the one-way parameters of the two-ports ?
5. What parameters of the two-ports are called primary, secondary?
6. What are the characteristic parameters of the two-ports?
7. What are the simplest two-ports and how to connect them to then?

# 7. Electric filters

## 7.1. General information about filters. Definitions and classification

Electric filters were proposed at the end of the nineteenth century and since then they have found application in virtually all electrical and electronic devices.

The theory of filters is represented by classical and modern filter theory.

The classical theory is based on the application of characteristic parameters of the four - terminal network,(two-port) that is, it involves matching the load with the parameters of the filter, which is practically difficult to perform. Therefore, after calculating the terms of the agreement, experimentally specify. Classical theory does not provide optimal results, although it requires minimal time and effort.

The modern theory of filters allows you to calculate optimal filters with high accuracy. It implies a preliminary approximation of the frequency characteristics of the filters by rational transmitting functions and further synthesis of circles for the implementation of these functions.

Consider the classical theory of filters.

An electric filter is called a two–port, which passes without attenuation signals with frequencies present in the bandwidth and holds signals with frequencies outside of this band (in the band of attenuation).

Cut-off frequencies are frequencies at the boundary of the bandwidth.

By location of the bandwidth distinguish:

a) a low pass filter (LPF) (Fig. 7.1,a)

It's bandwidth is $0 \le \omega \le \omega_c$, where $\omega$ – the frequency of the signal, transmitted without damping, $\omega_c$ – cut-off frequency – (limit frequency of bandwidth).

The dotted line in Fig. 7.1 shows real amplitude-frequency characteristics (AFC) $K(\omega)$;

b) a high-pass filter (HPF) (Fig. 7.1; b), it's bandwidth $\omega \ge \omega_c$;

c) band pass filter (BPF) (Fig. 7.1, c); it’s bandwidth $\omega_{c1} \le \omega \le \omega_{c2}$;

d) rejection filter (RF) (Fig. 7.1, d); it’s bandwidth $\omega_{c1} \ge \omega \ge \omega_{c2}$.

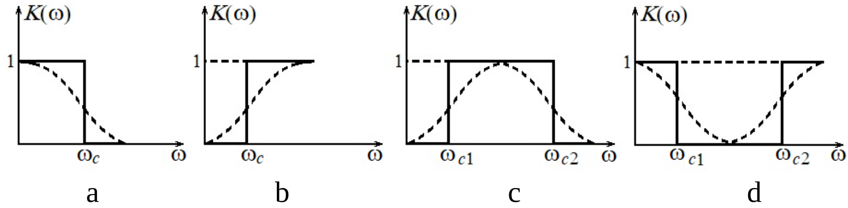



Fig. 7.1

### 7.2. General properties of characteristic filters parameters

Classical filter theory considers circuits composed entirely of reactive elements, that is, without taking into account losses, which leads to errors, since in real schemes there are always losses in the coils of inductance, capacitors, connecting conductors. These losses in the design of filters try to minimize.

If the filter is a four-pole, then the characteristic parameters are used for its description, in particular the characteristic transmission coefficient (5.50). Therefore, it can be assumed that the filter is a four-pole, which has a bandwidth of attenuation $\alpha = 0$ and in a non-pass band $\alpha \neq 0$. In the ideal filter in the non-pass band $\alpha \to \infty$.

In the classical theory for the construction of filters use symmetrical quadruple, which can be depicted in the form of П-like and T-like schemes replacement (Fig. 7.2).

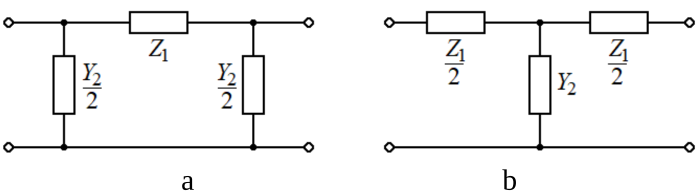



Fig.7.2

These schemes are the links on which the filters of the chain (step) structure are constructed (Fig. 7.3).

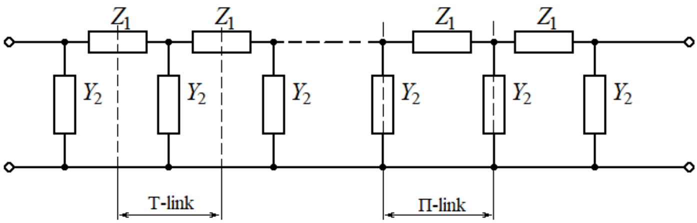



Fig. 7.3

This construction will be clear if one considers that each of the Π -like or T-like units can be divided into two Γ-like links connected cascade (Fig. 7.4, 7.5).

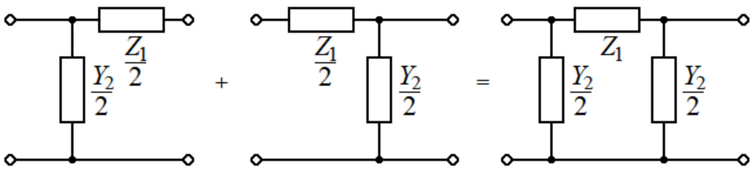



Fig. 7.4

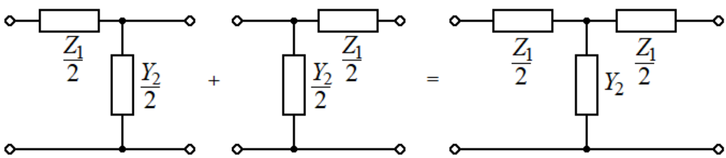



Fig. 7.5

As already noted, one-way parameters of symmetric quadrupole poles

$$Z_x = \frac{1}{Y_x}; \quad Z_k = \frac{1}{Y_k}. \tag{7.1}$$

With purely reactive filter elements

$$Z_x = jx_x, Z_k = jx_k. \tag{7.2}$$

Then the characteristic impedance of the filter by the expression (5.64)

$$Z_c = \sqrt{Z_x Z_k} = \sqrt{-x_x x_k}. \tag{7.3}$$

That is, the characteristic resistance of the filter is an actual value, if they have different signs (different in the nature of reactivity) and the imaginary magnitude, if they have the same signs.

Taking into account the formulas (7.2), the characteristic transmission coefficient of the filter corresponding to the expression (5.57) has the form

$$\gamma = \frac{1}{2}\ln\frac{1+\sqrt{\frac{Z_k}{Z_x}}}{1-\sqrt{\frac{Z_k}{Z_x}}} = \frac{1}{2}\ln\frac{1+\sqrt{\frac{x_k}{x_x}}}{1-\sqrt{\frac{x_k}{x_x}}}.$$

Let's consider individual cases.

*Impedances $x_x$ and $x_k$ have the same signs*. Then $Z_c$ in formula (7.3) is imaginary. If so

$$e^{2\gamma} = e^{2(\alpha+j\beta)} = e^{2\alpha}e^{j2\beta} = \frac{1+\sqrt{\frac{x_k}{x_x}}}{1-\sqrt{\frac{x_k}{x_x}}},$$

Than module

$$e^{2\alpha} = \left|\frac{1+\sqrt{\frac{x_k}{x_x}}}{1-\sqrt{\frac{x_k}{x_x}}}\right|.$$

and

$$\alpha = \frac{1}{2}\ln\left|\frac{1+\sqrt{\frac{x_k}{x_x}}}{1-\sqrt{\frac{x_k}{x_x}}}\right|. \tag{7.4}$$

That is the module of the characteristic transmission coefficient is real and positive.

Argument

$$\beta = \frac{1}{2}\mathrm{Arg}\frac{1+\sqrt{\frac{x_k}{x_x}}}{1-\sqrt{\frac{x_k}{x_x}}} = l\frac{\pi}{2}, \tag{7.5}$$

where $l = 0, \pm 1, \pm 2, \ldots$

In the expression (7.5) the value under the sign Arg is purely valid and depending on the relation between $x_x$ and $x_k$ may be positive or negative. In the first case

$$\beta = \frac{1}{2}\mathrm{Arg}\, e^{2\gamma} = \frac{1}{2}\mathrm{arctg}\frac{\mathrm{Im}(e^{2\gamma})}{\mathrm{Re}(e^{2\gamma})} = 0.$$

In the second case, that is $\beta = \frac{1}{2}l\pi = l\frac{\pi}{2}$, it corresponds to the expression (7.5).

Thus, under the imaginary $Z_c$ have $\alpha \neq 0$, that is the same signs $x_x$ and $x_k$ correspond to the band suppression band of a filter.

Impedances $x_x$ and $x_k$ have the different signs. Then formula (7.3) is valid. Also, according to formulas (7.4) and (7.5) we get

$$\alpha = \frac{1}{2}\ln\left|\frac{1+\sqrt{\frac{\pm x_k}{\mp x_x}}}{1-\sqrt{\frac{\pm x_k}{\mp x_x}}}\right| = \frac{1}{2}\ln\left|\frac{1+j\sqrt{\frac{|x_k|}{|x_x|}}}{1-j\sqrt{\frac{|x_k|}{|x_x|}}}\right| = 0;$$

$$\beta = \frac{1}{2}\mathrm{Arg}\frac{1+\sqrt{\frac{\pm x_k}{\mp x_x}}}{1-\sqrt{\frac{\pm x_k}{\mp x_x}}} = \mathrm{arctg}\sqrt{\frac{|x_k|}{|x_x|}}.$$

Thus, if we really $Z_c$ we have $\alpha = 0$, that is different signs $x_x$ and $x_k$ correspond to the bandwidth of the filter.

Consequently, we can conclude that, the frequencies, at which they $Z_c$ change their true value on the imaginary, and the reactance's $x_x$ or $x_k$ individually change the sign, lie on the boundary of the bandwidth, that are the cut off frequencies.

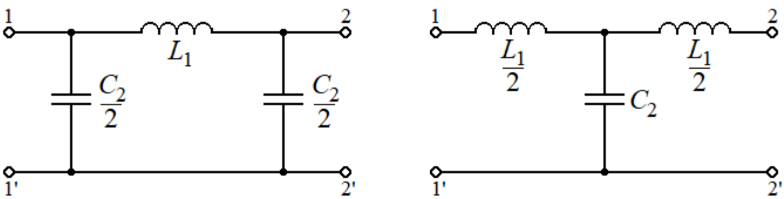
The general properties of the characteristic parameters of the filters are given in Table. 7.1

Table 7.1

| Signs $x_x$, $x_k$ | $Z_c$ | γ | α | | β | Band |
|---|---|---|---|---|---|---|
| different | Real: $Z_c = R_c$ | Imaginary | 0 | $x_x < 0$, $x_k > 0$ | $\text{arctg}\sqrt{\frac{\lvert x_k\rvert}{\lvert x_x\rvert}}$ | Transmission |
| | | | | $x_x > 0$, $x_k < 0$ | $-\text{arctg}\sqrt{\frac{\lvert x_k\rvert}{\lvert x_x\rvert}}$ | |
| equal | Imaginary: $Z_c = jx_c$ | Real or complex | $\frac{1}{2}\ln\left\lvert\frac{1+\sqrt{\frac{x_k}{x_x}}}{1-\sqrt{\frac{x_k}{x_x}}}\right\rvert$ | | $l\frac{\pi}{2}$, $l = 0, \pm1, \pm2$ | non-transmission |

## 7.3. Low - pass frequency filters

Let us consider П- and T-like links in which $Z_1$ is inductance and $Y_2$ – capacitance are taken (Fig. 7.6). We define one – sided parameters $Z_x$ and $Z_k$ for these schemes.




a b

Fig. 7.6

For the П - like scheme (Fig. 7.6, a), at idling terminal 2-2' we have

$$Z_{x\Pi}=\frac{\frac{1}{j\omega\frac{C_2}{2}}\left(j\omega L_1+\frac{1}{j\omega\frac{C_2}{2}}\right)}{\frac{1}{j\omega\frac{C_2}{2}}+j\omega L_1+\frac{1}{j\omega\frac{C_2}{2}}}=\frac{\frac{2}{j\omega C_2}\left(j\omega L_1+\frac{2}{j\omega C_2}\right)}{\frac{4}{j\omega C_2}+j\omega L_1}=$$

$$=-j\frac{2}{\omega C_2}\frac{2-\omega^2L_1C_2}{4-\omega^2L_1C_2}=-jx_{x\Pi}. \quad (7.6)$$

When short-circuiting the clamps 2-2' we have for the Π-like scheme

$$Z_{k\Pi}=\frac{\frac{1}{j\omega\frac{C_2}{2}}j\omega L_1}{\frac{1}{j\omega\frac{C_2}{2}}+j\omega L_1}=\frac{\frac{2L_1}{C_2}}{\frac{4}{j\omega C_2}+j\omega L_1}=j\frac{2\omega L_1}{2-\omega^2L_1C_2}=jx_{k\Pi}. \quad (7.7)$$

For a T-like scheme (Fig. 7.6, b), at idling terminal 2-2' we have for the T-like scheme

$$Z_{k\mathrm{T}}=j\omega\frac{L_1}{2}+\frac{1}{j\omega C_2}=-j\frac{2-\omega^2L_1C_2}{2\omega C_2}=-jx_{x\mathrm{T}}. \quad (7.8)$$

When short-circuiting the clamps 2-2' we have for the T-like scheme

$$Z_{k\mathrm{T}}=j\omega\frac{L_1}{2}+\frac{\frac{1}{j\omega C_2}j\omega\frac{L_1}{2}}{\frac{1}{j\omega C_2}+j\omega\frac{L_1}{2}}=j\omega\frac{L_1}{2}+\frac{\frac{L_1}{2C_2}j\omega 2C_2}{2-\omega^2L_1C_2}=$$

$$=j\omega\frac{L_1}{2}\left(1+\frac{2}{2-\omega^2L_1C_2}\right)=j\omega\frac{L_1}{2}\frac{4-\omega^2L_1C_2}{2-\omega^2L_1C_2}=jx_{k\mathrm{T}}. \quad (7.9)$$

As noted earlier, at the cut off frequencies, impedances $x_x$ or $x_k$ change their sign. Obviously, at the cut off frequencies, their values pass through zero, that is the cut off frequencies can be determined from the relations

$$x_{x\Pi}=0;\; x_{k\Pi}=0;\; x_{x\mathrm{T}}=0;\; x_{k\mathrm{T}}=0.$$

More than the obtained values determines is the cut off frequencies. From the expressions (7.6) - (7.9) it follows that the cut off frequencies can be determined from the ratio

$$x_{k\mathrm{T}} = 0,$$

that is

$$4 - \omega^2 L_1 C_2 = 0$$

where from

$$\omega_c = \frac{2}{\sqrt{L_1 C_2}}. \tag{7.10}$$

Determine the characteristic impedance of the LPF. Let's denote

$$R = \sqrt{Z_1 Z_2} = k,$$

where $Z_1 = \frac{1}{Y_1}$ and $Z_2 = \frac{1}{Y_2}$ - impedances, belonging to the Π–like and T-like skims of the filter sections. The value $R$ is called the nominal characteristic impedance of the filter. Filters, for which impedance $R = k = const$ and valid, are called "$k$" filters.

For the scheme in Fig.7.6

$$R = \sqrt{Z_1 Z_2} = \sqrt{j\omega L_1 \frac{2}{j\omega C_2}} = \sqrt{\frac{L_1}{C_2}}. \tag{7.11}$$

From the expressions (7.10) and (7.11) we find $L_1$ and $C_2$

$$L_1 = \frac{2R}{\omega_c}; C_2 = \frac{2}{\omega_c R}. \tag{7.12}$$

The characteristic impedance of the LPF is determined by the formula (7.3). For the Π-like filter scheme, we use the expressions (7.6) and (7.7)

$$Z_{c\Pi} = \sqrt{-x_{x\Pi} x_{k\Pi}} = \sqrt{\frac{2}{\omega C_2} \frac{2 - \omega^2 L_1 C_2}{4 - \omega^2 L_1 C_2} \frac{2\omega L_1}{2 - \omega^2 L_1 C_2}} =$$

$$= \sqrt{\frac{L_1}{C_2} \frac{4}{4 - \omega^2 L_1 C_2}}. \tag{7.13}$$

For the T-like filter scheme, using expressions (7.8) and (7.9), we get

$$Z_{c\text{T}} = \sqrt{-x_{x\text{T}}x_{k\text{T}}} = \sqrt{\frac{2-\omega^2 L_1 C_2}{2\omega C_2}\frac{\omega L_1}{2}\frac{4-\omega^2 L_1 C_2}{2-\omega^2 L_1 C_2}} =$$
$$= \sqrt{\frac{L_1}{C_2}\frac{4-\omega^2 L_1 C_2}{4}}. \quad (7.14)$$

Let substitute the expression (7.12) in the formulas (7.13) and (7.14). We have

$$Z_{c\Pi} = \frac{R}{\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}}; \quad Z_{c\text{T}} = R\sqrt{1-\left(\frac{\omega}{\omega_c}\right)^2}. \quad (7.15)$$

We introduce normalize characteristic impedances and frequencies

$$Z_{cn} = \frac{Z_c}{R}; \quad \omega_n = \frac{\omega}{\omega_c}. \quad (7.16)$$

Then we get the expression (7.15)

$$Z_{c\Pi n} = \frac{1}{\sqrt{1-\omega^2}}; \quad Z_{c\text{T}n} = \sqrt{1-\omega^2}. \quad (7.17)$$

You can see that

$$Z_{c\Pi n} Z_{c\text{T}n} = 1. \quad (7.18)$$

In Fig. 7.7

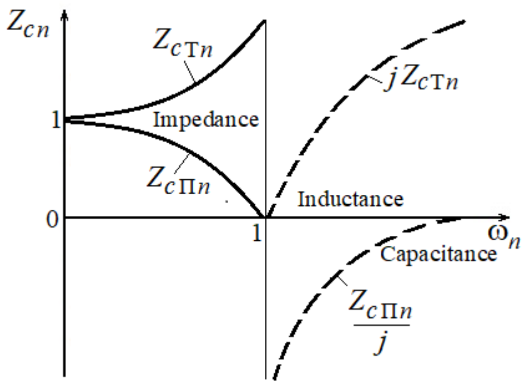



Fig. 7.7

In fig. 7.7 the formulas (7.17) construct graphs for the change of characteristic impedances $Z_{c\Pi n}$, $Z_{cTn}$ from frequency $\omega_n$. From the charts it is clear that the characteristic impedances $Z_c$ is very dependent on the frequency. Therefore it is impossible the load impedance matching with the characteristic impedance of the filter at all frequencies. Typically, such matching is achieved at the same frequency within the bandwidth.

Let’s defined attenuation and phase coefficient of low pass filter. In the formulas for determining $\gamma$, $\alpha$ and $\beta$ in the relation $\frac{x_k}{x_x}$ is present. We define this relation for the Π- and T-like sections of the filter. For a Π-like scheme, using the expressions (7.6) and (7.7), we get

$$\frac{x_{k\Pi}}{x_{x\Pi}} = \frac{-2\omega L_1}{2-\omega^2 L_1 C_2}\frac{\omega C_2}{2}\frac{4-\omega^2 L_1 C_2}{2-\omega^2 L_1 C_2} = \\ = -\omega^2 L_1 C_2 \frac{4-\omega^2 L_1 C_2}{(2-\omega^2 L_1 C_2)^2}. \tag{7.19}$$

For a T-like scheme, using expressions (7.8) and (7.9), we get

$$\frac{x_{kT}}{x_{xT}} = \frac{-\omega L_1}{2}\frac{4-\omega^2 L_1 C_2}{2-\omega^2 L_1 C_2}\frac{2\omega C_2}{2-\omega^2 L_1 C_2} = \\ = -\omega^2 L_1 C_2 \frac{4-\omega^2 L_1 C_2}{(2-\omega^2 L_1 C_2)^2}. \tag{7.20}$$

Compare with expressions (7.19) and (7.20), we get that α and β in Π- and T-like sections are defined by the same expressions. Using formulas (7.12), and go on to normalize values (7.16), we find

$$\frac{x_{k\Pi}}{x_{x\Pi}} = \frac{x_{kT}}{x_{xT}} = \frac{4\omega^2(\omega^2-1)}{(1-2\omega_n^2)^2}. \tag{7.21}$$

At bandwidth of LPF

$$0 \le \omega \le \omega_c \text{ or } 0 \le \omega_n \le 1.$$

At bandwidth of LPF (see Section 7.2) reactance $x_x$ and $x_k$ have different signs (compare expressions 7.6 – 7.9). Then (table. 7.1)

$$\begin{cases} \alpha = 0; \\ \beta = \operatorname{arctg}\sqrt{\left|\frac{x_k}{x_x}\right|} \end{cases}. \tag{7.22}$$

Using expression (7.21), we get

$$\beta = \text{arctg}\frac{2\omega_n\sqrt{\omega_n^2 - 1}}{1 - 2\omega_n^2} = \text{arctg}\frac{2\omega_n\sqrt{1 - \omega_n^2}}{(1 - \omega_n^2) - \omega_n^2} =$$

$$= \text{arctg}\frac{2\omega_n}{\sqrt{1 - \omega_n^2}}\frac{1}{1 - \dfrac{\omega_n^2}{1 - \omega_n^2}}. \tag{7.23}$$

Let's denote

$$\frac{\omega_n}{\sqrt{1 - \omega_n^2}} = \text{tg}\,\delta = \varphi.$$

Then from the expression (7.23)

$$\beta = \text{arctg}\frac{2\varphi}{1 - \varphi^2} = \text{arctg}\frac{2\,\text{tg}\,\varphi}{1 - \text{tg}^2\delta} = 2\delta = 2\,\text{arctg}\,\varphi =$$

$$= 2\,\text{arctg}\frac{\omega_n}{\sqrt{1 - \omega_n^2}} = 2\arcsin\omega_n.$$

That is, in the bandwidth of the LPF through the normalized values

$$\begin{cases} \alpha = 0; \\ \beta = 2\arcsin\omega_n. \end{cases} \tag{7.24}$$

In the band of non-transmission of the LPF

$$\omega_c \le \omega < \infty \text{ or } 1 \le \omega_n < \infty.$$

In the non-propagation band (see Section 7.2), the supports and have the same signs, as seen from the comparison of expressions (7.6) - (7.9). Then (Table. 7.1)

$$\begin{cases} \alpha = 0; \\ \beta = l\dfrac{\pi}{2}. \end{cases}$$

where $l = 0, \pm1, \pm2, \ldots.$

From expression (7.4), using (7.21), we get

$$\alpha = \frac{1}{2}ln\left|\frac{1 + \sqrt{\dfrac{x_k}{x_x}}}{1 - \sqrt{\dfrac{x_k}{x_x}}}\right| = \frac{1}{2}\ln\left|\frac{1 + \dfrac{2\omega_n\sqrt{\omega_n^2 - 1}}{2\omega_n^2 - 1}}{1 - \dfrac{2\omega_n\sqrt{\omega_n^2 - 1}}{2\omega_n^2 - 1}}\right| =$$

$$= \frac{1}{2}\ln\left|\frac{(\omega_n^2 - 1) + 2\omega_n\sqrt{\omega_n^2 - 1} + \omega_n^2}{(\omega_n^2 - 1) - 2\omega_n\sqrt{\omega_n^2 - 1} + \omega_n^2}\right| =$$

$$= \frac{1}{2}\ln\left|\frac{\left(\omega_n + \sqrt{\omega_n^2 - 1}\right)^2}{\left(\omega_n - \sqrt{\omega_n^2 - 1}\right)^2}\right| = \ln\left|\frac{\omega_n + \sqrt{\omega_n^2 - 1}}{\omega_n - \sqrt{\omega_n^2 - 1}}\right| =$$

$$= \ln\left|\frac{\left(\omega_n + \sqrt{\omega_n^2 - 1}\right)^2}{\omega_n^2 - (\omega_n^2 - 1)}\right| = \ln\left|\omega_n + \sqrt{\omega_n^2 - 1}\right| = 2\,\mathrm{arch}\,\omega_n.$$

The phase coefficient β in the non-transmissibility band can be determined by its value at the boundary of the non-transmissibility band, that is, at the cut off frequency $\omega_n = 1$. Then from the formula (7.24)

$$\beta = 2\arcsin\omega_n = 2\arcsin 1 = 2\frac{\pi}{2} = \pi \qquad (7.25)$$

If in the non-transmission band $\beta = l\frac{\pi}{2}$ (Table 7.1), then by the formula (7.25) $l = 2$.

Thus, in the band of rejection band of LPF through the normalized value

$$\begin{cases} \alpha = 2\,\mathrm{arch}\,\omega_n\,; \\ \beta = \pi. \end{cases} \qquad (7.26)$$

In fig. 7.8, the formulas (7.22) and (7.26) construct graphs for damping change α and the phase coefficient β from the frequency $\omega_n$.

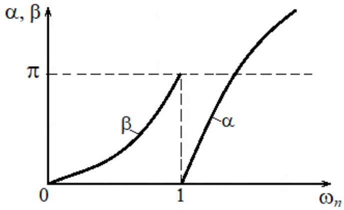


Fig. 7.8

The frequency characteristics $\alpha(\omega_n)$, $\beta(\omega_n)$, $Z_c(\omega_n)$, which schown in Fig. 7.7 and 7.8, are called the universes normalized characteristics of the LPF.

To increase the attenuation α in the of rejection band sections of LPF are connected in stage. Then accordance with the expression (7.24) in bandwidth

$$\begin{cases} \alpha_n = 0; \\ \beta_n = 2n \arcsin \omega_n . \end{cases}$$

In the rejection band sections of LPF in accordance with the expression (7.26)

$$\begin{cases} \alpha_n = 2n \operatorname{arch} \omega_n ; \\ \beta_n = n\pi. \end{cases} \tag{7.27}$$

where *n* is the number of cascaded connected sections of the LPF.

### 7.4. Derived filters like "*m*"

The task of improving the selectivity of the LPF, in particular, the increase of attenuation in the rejection band section can be resolved on the basis of modifiable schemes of filters – filters like "$m$". The prototype for filters like "$m$" is a filter of type "$k$". For reception of a filter like "$m$" is necessary in the scheme of the filter of type "$k$" successive inductance $L_1$ in the Π-like scheme to replace by parallel connection $LC$ circuit and parallel capacitance $C_2$ in T-like scheme – by sequential oscillation $LC$ circuit. The values of inductances and capacitance are chosen according to Fig. 7.9, where $L_1$ and $C_2$ are respectively the inductance and capacitance of a filter of type "$k$" (see Figure 7.6).

The coefficient "$m$" lies within $0 \leq m \leq 1$. It is seen that for $m = 1$, the filter type "$m$" is converted into a filter of type "$k$".

For sections of the filter type "$m$", the cut off frequency $\omega_c$ and characteristic impedance remain equal to these values $Z_c$ in the prototype (filter of type "$k$"). In Fig. 7.9 it is seen that the longitudinal branch in the Π-like scheme is a parallel oscillatory circuit, whose resonant frequency $\omega_{\infty\Pi}$ is defined by the expression

$$\omega_{\infty\Pi} = \frac{1}{\sqrt{mL_1 \dfrac{1-m^2}{4m} C_2}} = \frac{1}{\sqrt{1-m^2}} \frac{2}{\sqrt{L_1 C_2}} = \frac{1}{\sqrt{1-m^2}} \omega_c. \tag{7.28}$$

From Fig. 7.9, b it is seen that the transverse branch in the T-like scheme is a sequential oscillatory circuit $C$, whose resonant frequency $\omega_{\infty\mathrm{T}}$ is determined by the expression

$$\omega_{\infty T} = \frac{1}{\sqrt{\frac{1-m^2}{4m} L_1 m C_2}} = \frac{1}{\sqrt{1-m^2}} \frac{2}{\sqrt{L_1 C_2}} = \frac{1}{\sqrt{1-m^2}} \omega_c. \quad (7.29)$$

At the frequency $\omega_{\infty\Pi}$ impedance of the transverse branch in the П-like scheme tends to ∞, that is $\alpha \to \infty$. At the frequency $\omega_{\infty T}$ impedance of the transverse branch in the T-like scheme tends to 0, that is attenuation $\alpha \to \infty$ .

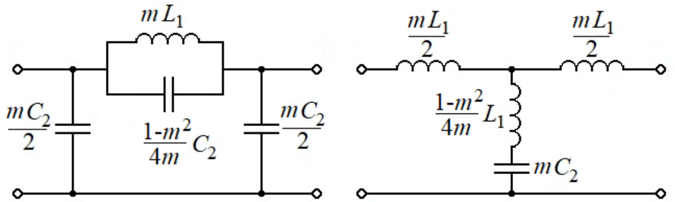



Fig. 7.9

From expressions (7.28) and (7.29) it is shown, that $\omega_{\infty\Pi} > \omega_c$, $\omega_{\infty T} > \omega_c$ ,that at frequencies are more cat off frequency, attenuation of the filters sharply increase and selectance increase too.

Graphics of change α from $\omega_n$ are shown in fig. 7.10, where from one can see that attenuation α in filter type "*m*" after frequencies $\omega_{\infty\Pi}$, $\omega_{\infty T}$ decries, approximating to the same ending meaning:

$$\alpha_\Pi = \ln\frac{1+m_\Pi}{1-m_\Pi}; \; \alpha_T = \ln\frac{1+m_T}{1-m_T}.$$

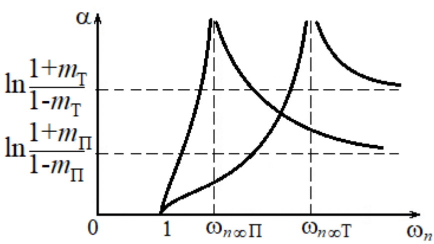



Fig. 7.10

Consider the characteristic impedance for a filter type "*m*".

In order to best match the load with the filter it is necessary that the characteristic impedance of the filter is as possible invariable within the bandwidth.

Let's break each of every part of the filter type "*m*" (see Fig. 7.9) into a cascade connection of two half – link (Fig. 7.11 and Fig. 7.12).

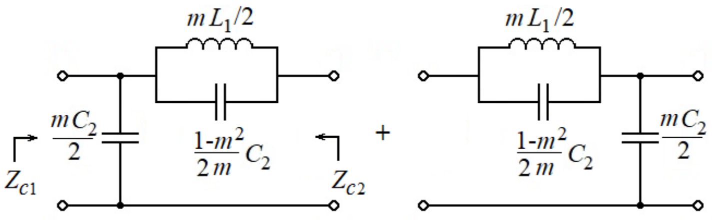



Fig. 7.11

Characteristic impedance $Z_{c1}$ to the left of the П-like (Fig. 7.11) and T-like (Fig.7.12) half - links, as the analysis shows, equal to the characteristic impedance $Z_c$ of the full link

$$Z_{C_1} = Z_C$$

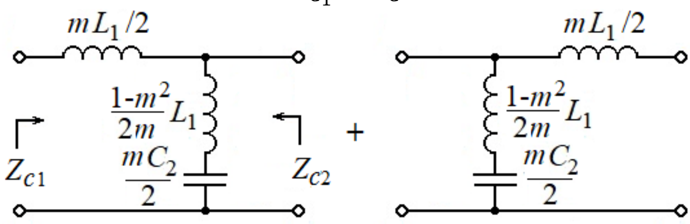



Fig. 7.12

Determine the characteristic impedance $Z_{c2}$ of the half - links to the right. According to Fig. 7.11

$$x_{x\Pi 2} = \frac{\dfrac{1}{j\omega\dfrac{1-m^2}{2m}C_2} j\omega\dfrac{mL_1}{2}}{\dfrac{1}{\dfrac{1}{j\omega\dfrac{1-m^2}{2m}C_2}} + j\omega\dfrac{mL_1}{2}} + \frac{1}{j\omega\dfrac{mC_2}{2}} = \tag{7.30}$$

$$= \frac{2j\omega L_1}{4-\omega^2(1-m^2)L_1C_2} + \frac{2}{j\omega m C_2} = -j\frac{2(4-\omega^2 L_1 C_2)}{\omega m C_2[4-\omega^2(1-m^2)L_1C_2]}.$$

$$x_{k\Pi 2} = \frac{\dfrac{1}{j\omega\dfrac{1-m^2}{2m}C_2} j\omega\dfrac{mL_1}{2}}{\dfrac{1}{j\omega\dfrac{1-m^2}{2m}C_2} + j\omega\dfrac{mL_1}{2}} = j\frac{2m\omega L_1}{4-\omega^2(1-m^2)L_1C_2}. \quad (7.31)$$

Then, using formula (7.3), using expressions (7.30) and (7.31), we get

$$Z_{c\Pi 2} = \sqrt{-x_{x\Pi 2}x_{k\Pi 2}} =$$

$$= \sqrt{\frac{2(4-\omega^2L_1C_2)}{\omega m C_2[4-\omega^2(1-m^2)L_1C_2]}\frac{2m\omega L_1}{4-\omega^2(1-m^2)L_1C_2}} =$$

$$= \sqrt{\frac{L_1}{C_2}\frac{4(4-\omega^2L_1C_2)}{[4-\omega^2(1-m^2)L_1C_2]^2}}.$$

Taking into account the expressions (7.12) and passing to the normalized quantities (7.16), we have

$$Z_{c\Pi 2n} = \frac{\sqrt{1-\omega_n^2}}{1-(1-m^2)\omega_n^2}.$$

From expression (7.18)

$$Z_{cT2n} = \frac{1}{Z_{c\Pi 2n}} = \frac{1-(1-m^2)\omega_n^2}{\sqrt{1-\omega_n^2}}.$$

In fig. 7.13 and 7.14 dependencies $Z_{c\Pi 2n}$ and $Z_{cT2n}$ from frequency $\omega_n$ are constructed. It turns out that at $m \approx 0{,}6$ the characteristic impedance $Z_{c2}$ in a large part of the bandwidth does not change. This allows you to fulfill the condition for the matching of the load with the parameters of the filter.

Half link filter type "*m*" can be used with links of type "*k*", which allows to take advantage of both one and second type of filters.

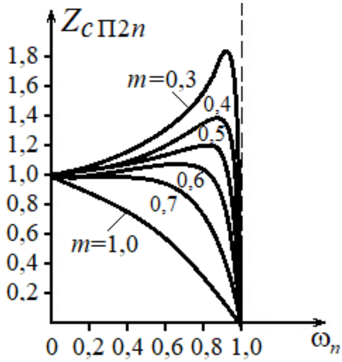



Fig. 7.13

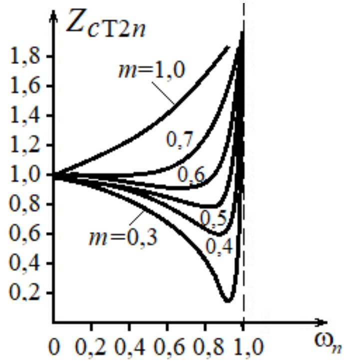



Fig. 7.14

### 7.5. Normalization of frequencies and impedances

Let LPF be given with cat off frequency $\omega_c$. In general, impedance of any branch of the filter is defined by the expression

$$Z(j\omega) = r + j\omega L + \frac{1}{j\omega C}.$$

It is necessary to determine the elements of the filter with cutoff frequency $K_\omega\omega$, where $K_\omega$ – the scale factor of frequency.

Increasing the cut-off frequency in $K_\omega$ times corresponds to the increase of each point of the abscissa of the frequency response of the filter in $K_\omega$ times when the magnitude of the ordinate and the general type of frequency characteristic are unchanged. Then the impedance of any branch of the new filter

$$Z(j\omega) = r + j\frac{\omega}{K_\omega}L + \frac{1}{j\frac{\omega}{K_\omega}C} = r + j\omega\frac{L}{K_\omega} + \frac{1}{j\omega\frac{C}{K_\omega}}. \qquad (7.32)$$

that is inductance and capacity of the filter should decrease in $K_\omega$times.

Thus, the connection between the parameters of both filters has the form

$$r_\omega = r;\;\; L_\omega = \frac{L}{K_\omega};\;\;\; C_\omega = \frac{C}{K_\omega}.$$

Let filter with a cut off frequency ω з is given. It is necessary to define the elements of the filter with the same cut off frequency $\omega_c$, but with the impedances in each branch in $K_c$ times greater, where $K_c$ – the scale factor of impedance:

$$Z_c(j\omega) = K_c r + jK_c\omega L + \frac{1}{j\frac{\omega}{K_c}C} = K_c r + j\omega K_c L + \frac{1}{j\omega\frac{C}{K_c}},$$

that is the impedance and inductance of the branch should increase in $K_c$ times, and the capacitance decrease in $K_c$ times. Thus, the relationship between the parameters of both filters has the form:

$$r_c = K_c r;\ L_c = K_c L;\ C_c = \frac{C}{K_c}.$$

If in common case filter with cat off frequency ω з transform to filter with cat off frequency $K_\omega$ω з and with impedances of each branch in $K_c$ times greater, then parameters of such filter are

$$r_2 = K_c r_1;\ L_2 = \frac{K_c}{K_\omega}L_1;\ C_2 = \frac{C_1}{K_c K_\omega}, \tag{7.33}$$

where $r_1$, $L_1$, $C_1$, $r_2$, $L_2$, $C_2$ - the parameters of the first and second filters, respectively.

At

$$K_c = \frac{1}{\omega_c} = \frac{1}{r_{l1}}$$

the cut off frequency of the converted filter $\omega_{c2} = K_c\ \omega_{c1} = 1$ and the resistance of its load $r_{l2} = K_c\ r_{l1} = 1$.

Such filter is called normalized. First, calculate for the normalized filter, get the value of the filter parameters, and then, through scale factors, pass to the actual values of the filter elements.

**Example 7.1**.

Find the value $L_1$ and $C_2$ for a single-link normalized ($\omega_c = 1$, $r_l = 1$) LPF of type "$k$".

*Solution*.

Select the load resistance $r_l$ equal to the nominal characteristic resistance $R = k$:

$$r_l = R = 1\ \text{Ohm}.$$

Parameters and the filter $L_1$ and $C_2$ will be determined by the expressions (7.12):

$$L_{1n} = \frac{2R}{\omega_c} = \frac{2\cdot 1}{1} = 2\text{H};\ \ C_{2n} = \frac{2}{\omega_c R} = \frac{2}{1\cdot 1} = 2\text{F}.$$

The diagram of the filter corresponds to Fig.7.6, b, where

$$\frac{L_1}{2} = 1\text{H};\; C_2 = 2\text{F}.$$

**Example 7.2.**

Calculate the LPF according to the following data.

$$f_c = 5 \text{ kHz}; \quad r_l = 1 \text{ k}\Omega.$$

*Solution.*

Parameters of normalized LPF:

$$\omega_{cn} = 1;\; r_{ln} = 1.$$

We find scale coefficients of frequency $K_\omega$ and impedance $K_c$:

$$K_\omega = \frac{\omega_c}{\omega_{cn}} = \frac{2\pi \cdot 5 \cdot 10^3}{1} = 3{,}14 \cdot 10^4;$$

$$K_c = \frac{r_l}{r_{ln}} = \frac{1000}{1} = 10^3.$$

In Example 7.1 the parameters of the normalized LPF were found: $L_{1n} = 2\text{H}$, $C_{2n} = 2\text{F}$.

Therefore, in this scheme, according to the formulas (7.33) we get

$$L_1 = \frac{K_C}{K_\omega} L_{1n} = \frac{10^3}{3.14 \cdot 10^4} \cdot 2 = 63.6 \cdot 10^{-3} \text{ H};$$

$$C_2 = \frac{1}{K_c K_\omega} C_{2n} = \frac{1}{10^3 \cdot 3.14 \cdot 10^4} \cdot 2 = 0.0637 \cdot 10^{-6}\text{F}.$$

Circuit of LPF is shown in fig. 7.15 (a – T-like, b – П-like).

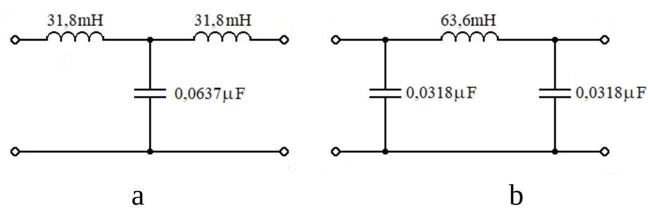



Fig. 7.15

**Example 7.3.**

Calculate the LPF according to the following data:

$$\omega_c = 3000\frac{\text{rad}}{\text{s}},\; R_n = 100 \text{ Ohms},\; Z_c(\omega_c) = 0{,}50 \text{ dB}.$$

at the frequency $2\omega_c$.

*Solution.*

The characteristic impedance at the cut off frequency $Z_c(\omega_c) = 0$ has T-like link of the filter (Fig. 7.7). Therefore, for the base normalized link, choose a T-like link for which $Z_{c\text{T}n}(1) = 0$. Determine the required number of links to provide the required attenuation in the band non-transmission. From formula (7.27)

$$\alpha_n(2\omega_c) = 2n\text{arch}\,2 \geq \frac{50}{8.686}\text{Nep},$$

that is

$$n \geq \frac{\alpha_n(2\omega_c)}{2\text{arch}2} = \frac{50}{8.686 \cdot 1.32} = 2.18.$$

Take $n = 3$.

We find scale factors. For $K_c$ at $R = 1$ we get

$$K_c = \frac{r_l}{R} = \frac{100}{1} = 100. \tag{7.34}$$

For $K_\omega$ by $\omega_n = 1$

$$K_\omega = \frac{\omega_c}{\omega_n} = \frac{3000}{1} = 3000. \tag{7.35}$$

Now we get the filter parameters according to formula (7.33) with the parameters of the base normalized filter (example 7.2) $L_{1n} = 2\text{H}$, $C_{2n} = 2\text{F}$:

$$L_1 = \frac{K_c}{K_\omega} L_{1n} = \frac{100}{3000} \cdot 2 = 66.6\text{H}; \tag{7.36}$$

$$C_2 = \frac{1}{K_c K_\omega} C_{2n} = \frac{2}{100 \cdot 3000} = 6.7\ \mu\text{F}. \tag{7.37}$$

The scheme of the filter is shown in Fig. 7.16

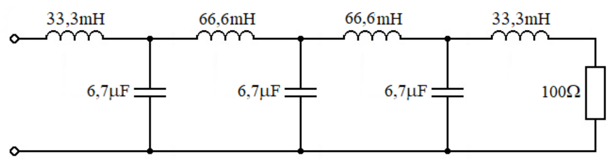



Fig. 7.16

The examined examples show the high efficiency of the normalization of the filters.

## 7.6 Frequency transformation

Calculated formulas and graphs obtained for LPF can be used to calculate filters of another type by the method of reception, which is called transformation or frequency. The essence of this method is that the imaginary frequency $j\omega$ in the expressions for the LPF is replaced by a certain imaginary value $\mathrm{T}(j\Omega) = j\mathrm{T}(\Omega)$, which is a function of another frequency. After such a change, the bandwidth on the characteristic of the LPF is converted into one or more other bandwidths, which corresponds to the characteristics of the new filter type.

Thus, the task is to determine the desired function $\mathrm{T}(j\Omega)$. The simplest frequency conversion has already been applied (see section 7.5) in the expression (7.32) when changing the frequency $\omega$ to $\omega/K_\omega$.

That is

$$j\omega = T(j\Omega) = jT(\Omega) = jT(K_\omega \omega).$$

In this case, the LPF with the cu off frequency ω was transformed into a low-pass filter with a cutoff frequency $\Omega_c = K_\omega \omega_c$.

It should be noted that according to Euler's formulas, any frequency corresponds to two imaginary frequencies $j\omega$ and $-j\omega$ on the complex plane. In terms of mathematics, positive and negative frequencies are equal. From the physical point of view, both values correspond to the concept of frequency oscillations.

## 7.7. High-pass filters

To obtain relations related to high pass filter, we use the method of frequency transformation. Let

$$j\omega = \mathrm{T}(j\Omega) = \frac{1}{j\Omega}.$$

Then the inductance impedance

$$j\omega L = \frac{1}{j\Omega} L = \frac{1}{j\Omega \frac{1}{L}} = \frac{1}{j\Omega C_\mathrm{e}}$$

becomes to impedance of the capacity, the value of which

$$C_e = \frac{1}{L}.$$

Capacity impedance

$$\frac{1}{j\omega C} = \frac{1}{\frac{1}{j\Omega}C} = j\Omega\frac{1}{C} = j\Omega\, L_e$$

becomes to impedance of the inductance, the value of which

$$L_e = \frac{1}{C}.$$

The ratio (7.1) for the LPF bandwidth

$$0 \le \omega \le \omega_c$$

becomes to relationship

$$0 \le \frac{1}{\Omega} \le \frac{1}{\Omega_c}$$

or

$$\Omega \ge \Omega_c. \tag{7.38}$$

The relation (7.38) corresponds to the high pass filter. For normalized LPF with $\omega_c = 1$ we have HPF with

$$\Omega_c = \frac{1}{\omega_c}, \tag{7.39}$$

that is also normalized by high pass filter.

Consequently, for a normalized low-pass filter into high- pass filter it is necessary to replace the inductance of HPF to capacitance in the LPF, and the capacitance to inductance:

$$L_h = \frac{1}{C_l}\ ; \quad C_h = \frac{1}{L_l}\ . \tag{7.40}$$

That is, for example, for an T-like link, the inductance $\frac{L_1}{2}$ must be replaced by the capacitance $\frac{2}{L_1}$, and the capacitance $C_2$ is replaced by the inductance $\frac{1}{C_2}$ (Fig. 7.17).

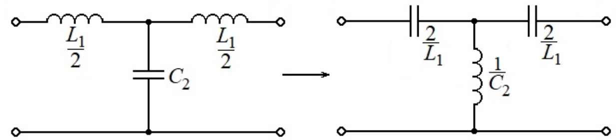



Fig.7.17

The cut off frequency $\Omega_c$ for the high-frequencyfilter is found from the formula (7.10) for the replacement $\omega_c$ of the LPF to $\frac{1}{\Omega_c}$ by the expression (7.39), and $L_1$ to $\frac{1}{C}$, $C_2$ to $\frac{1}{L}$ on the expression (7.40):

$$\frac{1}{\Omega_c} = \frac{1}{\sqrt{\frac{1}{C}\frac{1}{L}}}; \ \Omega_c = \frac{1}{2\sqrt{LC}}.$$

**Example 7.4.**

Performing a normalized LPF into LPF with a cutoff frequency $f_C$ = 5 kHz and resistance load $r_l$ = 100 Ohms.

*Solution*.

Parameters of normalized low-pass filter (example 7.1):

$$\frac{1}{2}L_{1nl} = 1\text{H}; \quad C_{2nl} = 2\text{F}.$$

The normalized HPF will have the following parameters (7.40)

$$L_{1nh} = \frac{1}{C_{2nl}} = \frac{1}{2}\text{H}; \quad C_{2nh} = \frac{1}{0.5L_{1nl}} = 1\text{F}.$$

The scale factors by $R = 1$, $\omega_n = 1$ for the formulas (7.34) and (7.35) are equal to:

$$K_c = \frac{r_l}{R} = \frac{100}{1} = 100; \ K_\omega = \frac{\omega_c}{\omega_n} = \frac{2\pi \cdot 5000}{1} = 31{,}4 \cdot 10^3.$$

Then for the desired filter for the expressions (7.36) and (7.37) we obtain:

$$L_h = \frac{K_c}{K_\omega}L_{1nh} = \frac{1}{31{,}4 \cdot 10^3} \cdot \frac{1}{2} = 1{,}59 \text{ mH}.$$

$$C_h = \frac{1}{K_c K_\omega}C_{2nh} = \frac{1}{100 \cdot 31{,}4 \cdot 10^3} \cdot 1 = 0{,}318 \ \mu\text{F}.$$

The diagram of the filter is shown in Fig. 7.18.

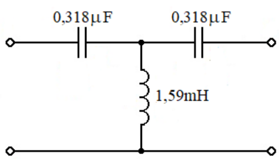



Fig. 7.18

**Example 7.5**.

Convert the "*m*" type LPF (see Fig. 7.9, b) into the "*m*" type high-fi filter.

Parameters of elements in longitudinal branches of the high-frequency electric field are obtained by the formulas (7.40):

$$C_h = \frac{1}{L_l} = \frac{1}{\frac{mL_1}{2}} = \frac{2}{mL_1}.$$

By the formula (7.40) we find the parameters of the transverse branch:

$$L_l = \frac{1}{C_h} = \frac{1}{\frac{1-m^2}{4m}L_1} = \frac{4m}{(1-m^2)L_1}.$$

The diagram of the filter is shown in Fig. 7.19.

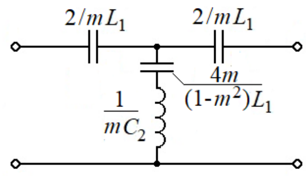



Fig. 7.19

**7.8. Band pass filter**

Let the transformation equation for band pass filter have the next form by the transformation frequency method

$$j\omega = \mathrm{T}(j\Omega) = \frac{\Omega_0^2 - \Omega^2}{j\Omega\Pi}, \tag{7.41}$$

where $\Omega_0$ – the geometric mean of the cut off frequence $\Omega_{c1}$ and $\Omega_{c2}$ of bandpass filter (BPF), and $\Pi$ – is the bandwidth of the BPF.

$$\Omega_0 = \sqrt{\Omega_{c1}\Omega_{c2}}; \tag{7.42}$$

then

$$\Pi = \Omega_{c1} - \Omega_{c2}. \tag{7.43}$$

Let the output LPF have a normalized cut off frequency $\omega_c = 1$. Two imaginary frequencies correspond to the complex plane of this frequency

$$j\omega = j\omega_c = j \cdot 1 = j; \quad j\omega = -j\omega_c = -j \cdot 1 = -j. \tag{7.44}$$

Then, substituting the formulas (7.44) in (7.41), we obtain (at $j\omega = j$)

$$j = \frac{\Omega_0^2 - \Omega_{c1}^2}{j\Omega_{c1}\Pi}$$

or

$$\Omega_{c1}^2 - \Pi\Omega_{c1} - \Omega_0^2 = 0.$$

Hence

$$(\Omega_{c1})_{1,2} = \frac{\Pi}{2} \pm \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}.$$

Interest have only positive frequency. Therefore

$$\Omega_{c1} = \frac{\Pi}{2} + \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}.$$

At

$$j\omega = -j, \qquad -j = \frac{\Omega_0^2 - \Omega_{c2}^2}{j\Omega_{c2}\Pi}$$

or

$$\Omega_{c2}^2 + \Pi\Omega_{c2} - \Omega_0^2 = 0.$$

Hence

$$(\Omega_{c2})_{1,2} = -\frac{\Pi}{2} \pm \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}.$$

Positive frequency $\Omega_{c2}$ are defined are defined by expression

$$\Omega_{c2} = -\frac{\Pi}{2} + \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}.$$

Now, using the transform (7.41), the complex impedance of the inductance

$$j\omega L = \frac{\Omega_0^2 - \Omega^2}{j\Omega\Pi} L = \frac{\Omega_0^2 L}{j\Omega\Pi} - \frac{\Omega^2 L}{j\Omega\Pi} = \frac{1}{j\Omega\frac{\Pi}{\Omega_0^2 L}} + j\Omega\frac{L}{\Pi} = \frac{1}{j\Omega C_\mathrm{e}} + j\Omega L_\mathrm{e},$$

where

$$L_\mathrm{e} = \frac{L}{\Pi};\ C_\mathrm{e} = \frac{\Pi}{\Omega_0^2 L}. \tag{7.45}$$

That is, the inductance $L$ is converted into a serial connection of equivalent inductance $L_{\mathrm{e}}$ and equivalent capacitance $C_{\mathrm{e}}$.

For complex conductance of capacity $C$ we have

$$j\omega C = \frac{\Omega_0^2 - \Omega^2}{j\Omega\Pi} C = \frac{\Omega_0^2 C}{j\Omega\Pi} - \frac{\Omega^2 C}{j\Omega\Pi} = \frac{1}{j\Omega \dfrac{\Pi}{\Omega_0^2 C}} + j\Omega \frac{C}{\Pi} = \frac{1}{j\Omega L_{\mathrm{e}}} + j\Omega C_{\mathrm{e}},$$

where

$$C_{\mathrm{e}} = \frac{C}{\Pi}; \;\; L_{\mathrm{e}} = \frac{\Pi}{\Omega_0^2 C}. \tag{7.46}$$

That is, the capacitance $C$ is converted into a parallel connection of inductance $L_{\mathrm{e}}$ and capacitance $C_{\mathrm{e}}$. Thus, the inductance $L$ is converted into a series oscillatory circuit, the capacitance $C$ – in the parallel oscillatory circuit.

Resonance frequency of the both circuits

$$\Omega_r = \frac{1}{\sqrt{L_{\mathrm{e}} C_{\mathrm{e}}}} = \frac{1}{\sqrt{\dfrac{L}{\Pi} \cdot \dfrac{\Pi}{\Omega_0^2 L}}} = \frac{1}{\sqrt{\dfrac{\Pi}{\Omega_0^2 C} \cdot \dfrac{C}{\Pi}}} = \Omega_0. \tag{7.47}$$

For example, for a T-like link of the low-pass filter we obtain the corresponding values of the band pass filter SF branches parameters (Fig. 7.20, a, b) by the formulas (7.45) and (7.46):

$$\begin{cases} L_{1\Pi} = \dfrac{L_1}{\Pi} = \dfrac{L_1}{2\Pi}; C_{1\Pi} = \dfrac{\Pi}{\Omega_0^2 L_{1l}} = \dfrac{2\Pi}{\Omega_0^2 L_1}; \\ L_{2\Pi} = \dfrac{\Pi}{\Omega_0^2 C_{2l}} = \dfrac{\Pi}{\Omega_0^2 C_2}; \;\; C_{2\Pi} = \dfrac{C_{2l}}{\Pi} = \dfrac{C_2}{\Pi}. \end{cases}$$

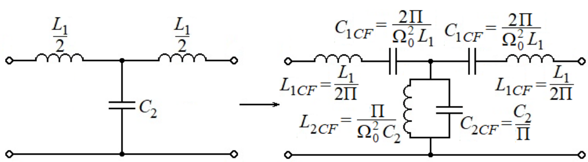



Fig. 7.20

**Example 7.6.**

Transform the normalized T-like LPF (Example 7.1) with the parameters $\frac{1}{2} L_{1nlpf} = 1\mathrm{H}$, $C_{2nlpf} = 2\mathrm{F}$ in the SF with $r_l = 75$ Ohm, the bandwidth $\Pi = 8 \cdot 10^{3}$ rad/s and the resonant frequency $\Omega_0 = 1.5 \cdot 10^6$ rad/s.

Let's find the parameters of the normalized SF by the formulas (7.45) and (7.46):

$$L_{1nCf} = \frac{L_{1l}}{\Pi} = \frac{1}{8 \cdot 10^3} = 0{,}125 \text{ mH};$$

$$C_{1nCf} = \frac{\Pi}{\Omega_0^2 L_{1l}} = \frac{8 \cdot 10^3}{(1.5 \cdot 10^6)^2 \cdot 1} = 3550 \text{ nF};$$

$$L_{2nCf} = \frac{\Pi}{\Omega_0^2 C_{2l}} = \frac{8 \cdot 10^3}{(1.5 \cdot 10^6)^2 \cdot 2} = 0{,}0018 \text{ μF};$$

$$C_{2nCf} = \frac{C_{2l}}{\Pi} = \frac{2}{8 \cdot 10^3} = 250 \text{ pF}.$$

Scale factor $K_c$ by $R = 1$ with formula (7.34) is

$$K_C = \frac{r_l}{R} = \frac{75}{1} = 75.$$

Then for the wanted SF

$$L_{1Cf} = K_C L_{1nCf} = 75 \cdot 0{,}125 \cdot 10^{-3} = 9{,}375 \text{ mH};$$

$$C_{1Cf} = \frac{1}{K_C} C_{1nCf} = \frac{1}{75} \cdot 3550 \cdot 10^{-12} = 47{,}3 \text{ pF};$$

$$L_{2Cf} = K_C L_{2nCf} = 75 \cdot 0{,}0018 \cdot 10^{-6} = 0{,}135 \text{ μH};$$

$$C_{2Cf} = \frac{1}{K_C} C_{2nCf} = \frac{1}{75} \cdot 250 \cdot 10^{-12} = 3{,}33 \text{ pF}.$$

### 7.9.Rejection filter RF

Let's by the transformation method have the form of the transformation equation is

$$j\omega = \mathrm{T}(j\Omega) = \frac{j\Omega\Pi}{\Omega_0^2 - \Omega^2}, \tag{7.48}$$

where $\Omega_0$ is determined by the formula (7.42), $\Pi$ – pass band of rejection filter (RF), is determined by the formula (7.43).

Analogically with the band pass filter, we solve the equation

$$j = \frac{j\Omega_{c1}\Pi}{\Omega_0^2 - \Omega_{c1}^2}$$

or

$$\Omega_{c1}^2 + \Pi\Omega_{c1} - \Omega_0^2 = 0.$$

From here

$$(\Omega_{c1})_{1,2} = -\frac{\Pi}{2} \pm \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}.$$

For positive frequencies

$$\Omega_{c1} = -\frac{\Pi}{2} + \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}.$$

Let's solve the equation

$$-j = \frac{j\Omega_{c2}\Pi}{\Omega_0^2 - \Omega_{c2}^2}$$

or

$$\Omega_{c2}^2 - \Pi\Omega_{c1} - \Omega_0^2 = 0.$$

Hence

$$(\Omega_{c2})_{1,2} = \frac{\Pi}{2} \pm \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}$$

For positive frequencies

$$\Omega_{c1} = \frac{\Pi}{2} + \sqrt{\frac{\Pi^2}{4} + \Omega_0^2}$$

That is in comparison with the band pass filter BPF, the boundary frequencies changed places.

Now, using the transformation (7.48), we obtain the complex conductivity of the inductor:

$$\frac{1}{j\omega L} = \frac{\Omega_0^2 - \Omega^2}{j\Omega\Pi L} = \frac{\Omega_0^2}{j\Omega\Pi L} - \frac{\Omega^2}{j\Omega\Pi L} =$$

$$\frac{1}{j\Omega\frac{\Pi L}{\Omega_0^2}} + j\Omega\frac{1}{\Pi L} = \frac{1}{j\Omega L_{\mathrm{e}}} + j\Omega C_{\mathrm{e}}, \tag{7.49}$$

where

$$C_{\mathrm{e}} = \frac{1}{\Pi L}; \quad L_{\mathrm{e}} = \frac{\Pi L}{\Omega_0^2}.$$

That is, inductance $L$ is converted into a parallel connection of inductance $L_\text{e}$ and capacitance $C_\text{e}$.

For complex resistance of capacity

$$\frac{1}{j\omega C}=\frac{\Omega_0^2-\Omega^2}{j\Omega\Pi C}=\frac{\Omega_0^2}{j\Omega\Pi C}-\frac{\Omega^2}{j\Omega\Pi C}=$$

$$\frac{1}{j\Omega\frac{\Pi C}{\Omega_0^2}}+j\Omega\frac{1}{\Pi C}=\frac{1}{j\Omega C_\text{e}}+j\Omega L_\text{e}, \tag{7.50}$$

where

$$L_\text{e}=\frac{1}{\Pi C};\quad C_\text{e}=\frac{\Pi C}{\Omega_0^2},$$

that is, the capacitance $C$ is converted into a series connection and iinductance $L_\text{e}$ and capacitance $C_\text{e}$.

Thus, the inductance is converted into a parallel oscillatory circuit, and the capacitance $C$ is a series oscillatory circuit. The resonance frequency of both circuits coincides with the result (7.47) for the band filter BPF.

For example, for a T-like link of a low – pass filter, we obtain the corresponding values of the parameters of the branches of the rejection filter RF (Fig. 7.21) by the formulas (7.49) and (7.50):

$$\begin{cases} L_{1cf}=\dfrac{1}{\Pi L_{1l}}=\dfrac{2}{\Pi L_{1l}};\; C_{1cf}=\dfrac{\Pi L_{1l}}{\Omega_0^2}=\dfrac{\Pi L_1}{2\Omega_0^2}; \\ L_{2cf}=\dfrac{1}{\Pi C_{2l}}=\dfrac{1}{\Pi C_2};\; C_{2cf}=\dfrac{\Pi C_{2l}}{\Omega_0^2}=\dfrac{\Pi C_2}{\Omega_0^2}. \end{cases}$$

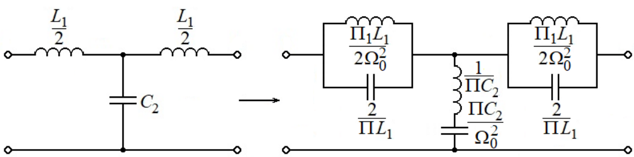
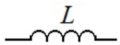
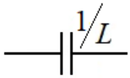
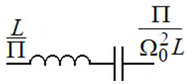
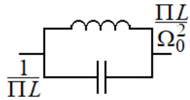
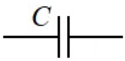
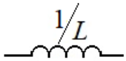
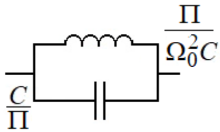
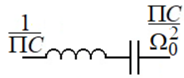



Fig.7.21

In table. 7.2 the correspondence between the elements of the LPF, HF, SF, and ZF, obtained by the method of frequency transformation, is indicated.

Table 7.2

| LBF | HBF | BF | RF |
|---|---|---|---|
|  |  |  |  |
|  |  |  |  |

### 7.10. Elements of filters synthesis

*Stages of filters synthesis for LPF*. Under the modern theory of filters, which involves the approximation of the frequency characteristics by the most suitable rational functions, it is possible to isolate the following steps to the synthesis of the thesis for the LPF.

1. The technical requirements for FFL LPF are formulated. The ideal frequency response at the cut off frequency is shown in Fig. 7.22 and is written as follows:

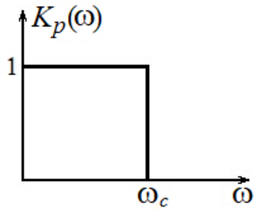



Fig.7.22

$$K_p(\omega) = \begin{cases} 1, & 0 \le \omega \le \omega_c; \\ 0, & \omega \ge \omega_c. \end{cases}$$

Here $K_p(\omega)$ is the ratio of power transmission.

In this case no requirements for phase-frequency characteristics (FHC) is not put. That is such a synthesis is a synthesis for a given AFC.

It is clear that the idealized frequency response for fig. 7.22 physically can not be realized and therefore the synthesis continues.

2. The idealized AFC for Fig. 7.22 is approximated by such a function, which follows it will be possible to realize in the physical circuit.

3. For the approximated frequency response, find the transfer function $K(p)$ – the dependence of the transmission coefficient on the operator $p = \sigma + j\omega$ on the complex plane. In comparison with the

complex function of the circuit $K(j\omega)$ as an argument is not the imaginary frequency $j\omega$, but the complex operator $p = \sigma + j\omega$, ie simply imaginary frequency $j\omega$, is replaced by the operator $p$.

4. Find the coordinates of the zeros and poles of the transfer function $K(p)$ and build the principle filter scheme for them.

*Approximation by the Butterworth filter*. The amplitude - frequency response for fig, 7.22 can be approximated by a filter with a maximum flat characteristic – a Butterworth filter. For him, the transmission factor of power

$$K_p(\omega_n) = \frac{1}{1+\omega_n^{2n}}, \tag{7.51}$$

where $\omega_n = \frac{\omega}{\omega_c}$ – normalized frequency, $n$- the order of the filter.

In fig. 7.23 shows AFC of Butterworth filter is shown at $n = 1$ and $n = 5$. It is clear, that $n$ is more, the more precisely the AFC is approximated for fig.7.22.

At the cut off frequency ($\omega_n = 1$), the transmission ratio of power $K_p = 0{,}5$.To estimate the signal attenuation, take a decimal logarithm from, i.e. $\alpha(\omega) = 10\lg K_p(\omega_н) = 10\lg 0.5 = -3.01\text{dB}$.

This value don't depends on the order of filter.

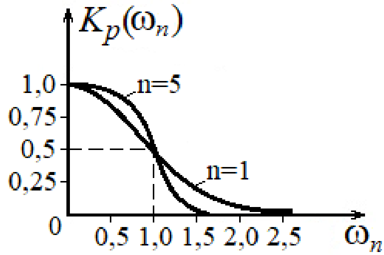



Fig.7.23

In the bandwidth of the LPF, when $\omega_n \gg 1$, we obtain from the formula (7.51)

$$K_p(\omega_n) \approx \omega_n^{-2n}.$$

Attenuation

$$\alpha(\omega) = 10\lg K_p(\omega_n) = -20n\lg\omega_n \text{dB}.$$

The rate of attenuation in the band of non-transmission is

$$\alpha(\omega) = -20n\lg 2 = -6n\frac{\text{dB}}{\text{octave}}.$$

That is, the increase in frequency twice gives an attenuation of 6 dB/octave. Octave is an interval of frequencies, the boundaries of which differ two time.

Determine the transfer function of the Butterworth filter.

Replace in the formula (7.51) $\omega_n$ with $j\omega_n$. If the transmission factor of power is

$$K_p(\omega_n) = K(j\omega_n) \cdot K(-j\omega_n),$$

where $K(j\omega_n) = \frac{\dot{U}_2}{\dot{U}_1}$ - the voltage transfer coefficient of two-pole, then the transfer function of the power $K_p(\omega_n)$ is a even and real number, that is, it does not take into account the phase proportions when the signal passes through the four-pole. Then, by the expression (7.51)

$$K_p(j\omega_n) = \frac{1}{1 + (j\omega_n)^{2n}} = \frac{1}{1 + (-1)^n(\omega_n)^{2n}}$$

Now propagate the action of function $K_p(j\omega_n)$ from the imaginary axis to the entire plane of complex frequencies. For this, replace $j\omega_n$ with $p_n = \sigma + j\omega_n$. Get it

$$K_p(p_n) = \frac{1}{1 + (-1)^n p_n^{2n}}.$$

Characteristic equation

$$1 + (-1)^n p_n^{2n} = 0 \qquad (7.52)$$

gives 2*n* poles on compex plane.

Now transfer function for Butterworth filter is written as

$$K(p_n) = \frac{1}{(p_n - p_{n1})(p_n - p_{n2}) \dots (p_n - p_{n2n})}. \qquad (7.53)$$

At $n = 1$

$$1 + (-1)^1 p_n^2 = 0,\ 1 - p_n^2 = 0,\ p_n^2 = 1.$$

Hence the roots

$$p_{n1} = 1, \quad p_{n2} = -1.$$

In fig. 7.24,a the location of these roots in the complex plane are shown.

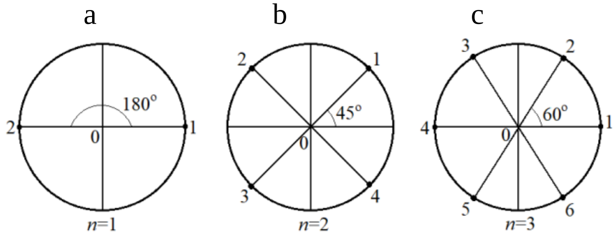



Fig.7.24

At $n = 2$

$$1 + (-1)^2 p_n^4 = 0, \qquad p_n^4 = -1.$$

Hence the roots

$$p_n^2 = \pm\sqrt{-1} = \pm j = \pm e^{j\frac{\pi}{2}};\ p_n = \pm\sqrt{\pm e^{j\frac{\pi}{2}}};$$

$$p_{n1} = +\sqrt{+e^{j\frac{\pi}{2}}} = e^{j\frac{\pi}{4}};$$

$$p_{n2} = +\sqrt{-e^{j\frac{\pi}{2}}} = +\sqrt{-e^{j\left(\frac{\pi}{2}+\pi\right)}} = +\sqrt{e^{j\frac{3\pi}{2}}} = e^{j\frac{3\pi}{4}};$$

$$p_{n3} = -\sqrt{e^{j\frac{\pi}{2}}} = -e^{j\frac{\pi}{4}} = e^{j\left(\frac{\pi}{4}+\pi\right)} = e^{j\frac{5\pi}{4}};$$

$$p_{n4} = -\sqrt{-e^{j\frac{\pi}{2}}} = -e^{j\frac{3\pi}{4}} = e^{j\left(\frac{3\pi}{4}+\pi\right)} = e^{j\frac{7\pi}{4}}.$$

In fig. 7.24,b the location of these roots in the complex plane is shown.

At $n = 3$

$$1 + (-1)^3 p_n^6 = 1 - p_n^6 = 0, \quad p_n^6 = -1.$$

Hence the roots

$$p_n^3 = \pm\sqrt{1},\ p_n^3 = 1,\ p_n^3 = -1 = j^2 = e^{j2\frac{\pi}{2}} = e^{j\pi};$$

$$p_{n1} = 1;$$

$$p_{n2} = \sqrt[3]{-1} = \sqrt[3]{j^2} = \sqrt[3]{e^{j2\frac{\pi}{2}}} = \sqrt[3]{e^{j\pi}} = e^{j\frac{\pi}{3}};$$

$$p_{n3} = \sqrt[3]{1} = \sqrt[3]{j^4} = \sqrt[3]{e^{j4\frac{\pi}{2}}} = \sqrt[3]{e^{j2\pi}} = e^{j\frac{2\pi}{3}};$$

$$p_{n4} = \sqrt[3]{-1} = -1;$$

$$p_{n5} = \sqrt[3]{1} = \sqrt[3]{j^8} = \sqrt[3]{e^{j8\frac{\pi}{2}}} = \sqrt[3]{e^{j4\pi}} = e^{j\frac{4\pi}{3}};$$

$$p_{n6} = \sqrt[3]{-1} = \sqrt[3]{j^{10}} = \sqrt[3]{e^{j10\frac{\pi}{2}}} = \sqrt[3]{e^{j5\pi}} = e^{j\frac{5\pi}{3}}.$$

In fig. 7.24,c the location of these roots in the complex plane is shown.

From Fig. 7.24, it is clear, that all poles are located at identical angles to each other, equal $\frac{\pi}{n}$. If $n$ – n odd, then $p_{n1} = 1$, if $n$ – twin, then $p_{n1} = e^{j\frac{\pi}{n}}$.

From Fig. 7.24 it is seen, that the picture of the location of the poles has quadrant symmetry, that is, with respect to the vertical axis, passing through the center, the picture is symmetric. Therefore, for the synthesis of a circuit, only those poles, located in the left on half-plane are taken (the roots of the characteristic equation, corresponding to these poles, have a negative real part,it is corresponding to the attenuation of processes and the presence in the implemented circle, of the active resistance). Mirror the image in the right half-plane is not taken into account (Fig. 7.24).

*Approximation by the Chebyshev filter*. Amplitude-frequency characteristic in Fig. 7.22 can be approximated with Chebyshev approximation – a Chebyshev filter.To do this, the transmission ratio is power

$$K_p(\omega_n) = \frac{1}{1+\varepsilon^2 \mathrm{T}_n^2(\omega_n)}, \tag{7.54}$$

where $\varepsilon < 1$ – the coefficient of non-uniformity of the characteristic in the bandwidth; $\mathrm{T}_n^2(\omega_n)$ – Chebyshev's polynomial of the *n*-order, which is determined by the formula

$$\mathrm{T}_n(x) = \cos(n \arccos x). \tag{7.55}$$

This polynomial has an important property: be $-1 < x < 1$ value of $\mathrm{T}_n(x)$ is the list deviates (comparatively with other polynomials) from zero.

At $|x| \gg 1$ values $\mathrm{T}_n(x)$ increase sharply. That is, the frequency characteristic for fig. 7.22 with a crestal peak (Fig. 7.25) is realized.

The function $\mathrm{T}_n(x)$ is determined from the recurrence ratio

$$\mathrm{T}_n(x) = 2x\mathrm{T}_{n-1}(x) - \mathrm{T}_{n-2}(x) \tag{7.56}$$

for $n = 0$ $\quad \mathrm{T}_0(x) = \cos 0 = 1,$

for $n = 1$ $\quad \mathrm{T}_1(x) = \cos(\arccos x) = x.$

This follows from the expression (7.53).

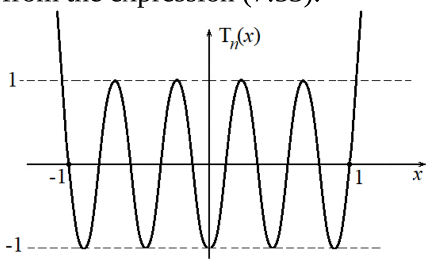



Fig.7.25

Let $n = 0$, then by the formula (7.54)

$$K_p(\omega_n) = \frac{1}{1+\varepsilon^2}.$$

Let $n = 1$, then by the formula (7.54)

$$K_p(\omega_n) = \frac{1}{1+\varepsilon^2\omega_n^2}.$$

In the bandwidth $0 \le \omega_n \le 1$, that is, $K_p(\omega_n)$ in the bandwidth range of LPF varies from 1 to $\frac{1}{1+\varepsilon^2}$.

Let it be $n = 2$, then by the formula (7.56)

$\mathrm{T}_2(x) = 2x\mathrm{T}_1(x) - \mathrm{T}_0(x) = 2x^2 - 1.$

Then according the formula (7.54) we get

$$K_p(\omega_n) = \frac{1}{1+\varepsilon^2 T_2^2(\omega_n)} = \frac{1}{1+\varepsilon^2(2\omega_n^2-1)^2}.$$

Let it be $n = 3$, then by the formula (7.56)

$$\mathrm{T}_3(x) = 2x\mathrm{T}_2(x) - \mathrm{T}_1(x) = 2x(2x^2-1) - x =$$
$$= 4x^3 - 2x - x = 4x^3 - 3x.$$

Now, by the formula (7.54)

$$K_p(\omega_n) = \frac{1}{1+\varepsilon^2\mathrm{T}_3^2(\omega_n)} = \frac{1}{1+\varepsilon^2(4\omega_n^3-3\omega_n)^2}.$$

У смузі пропускання $K_p(\omega_n)$ змінюється у межах від 1 до $\frac{1}{1+\varepsilon^2}$ etc.

That is, in the general case in the bandwidth the value $K_p(\omega_n)$ ranges from 1 to $\frac{1}{1+\varepsilon^2}$, if $\omega_n \gg 1$, that is, outside the bandwidth, the value $K_p(\omega_n)$ alls sharply.

Fig. 7.26 shows characteristic graphics of the frequency characteristics of the transmission coefficient for the Chebyshev filter at $n = 2$ and $n = 3$.

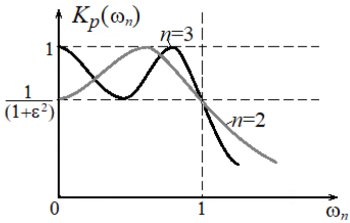



Fig. 7.26

It is seen that the magnitude of the ripples in the bandwidth depends on ε (increases with growth $\varepsilon$). To obtain the desired frequency characteristic, select a pair of parameters $\varepsilon$ and $n$.

Determine the transfer function of the Chebyshev filter. Replace in the formula (7.54) $\omega_n$ on $p_n = \sigma + j\omega_n$.

$$K_p(p_n) = \frac{1}{1 + \varepsilon^2 \mathrm{T}_n^2(p_n)}.$$

It's characteristic equation

$$1 + \varepsilon^2 \mathrm{T}_n^2(p_n) = 0. \tag{7.57}$$

The solution of the equation (7.57) is quite complex. The procedure for determining the roots of the equation (7.57) is as follows:

1) calculate the auxiliary parameter

$$a = \frac{1}{n}\mathrm{arch}\frac{1}{\varepsilon} = \frac{1}{n}\ln\left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}\right);$$

2) find the poles of the Butterworth filter in the same order,

3) abscissa each pole Chebyshev filter is found by multiplying the corresponding sha into Butterworth filter abscissa, ordinate each pole Chebyshev filter – as a product of the same cha into ordinate corresponding to each pole filter Botteworth,

4) using the coordinates pole Chebyshev filter, record Chebyshev filter transfer function similar ratio (7.53).

*Implementation of filters*. Consider the realization of the LPF. The order of the LPF is determined by the number of poles of the transfer function of the filter.

Consider the first-order filter. It is implemented in the first-order circuit in the form of a *RC*-four-pole (Fig. 7.27). For him

$$K(p) = \frac{U_{out}(p)}{U_{in}(p)} = \frac{\frac{1}{pC}}{R + \frac{1}{pC}} = \frac{1}{1 + pRC}. \tag{7.58}$$

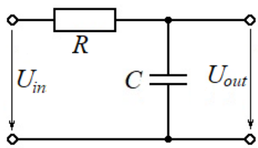



Fig. 7.27

Characteristic equation

$$1 + pRC = 0.$$

Its root

$$p_1 = -\frac{1}{RC} = -\frac{1}{\tau},$$

where $\tau = RC$ – is the time of the filter.

If specified $\tau = RC$, you can arbitrarily set $R$ or $C$.

Consider the second-order filter. It is implemented by the second order circuit in the form of a Γ-similar two-port (Fig.7.28).

For him

$$K(p) = \frac{U_{out}(p)}{U_{in}(p)} = \frac{\dfrac{R \cdot \frac{1}{pC}}{R + \frac{1}{pC}}}{pL + \dfrac{R \cdot \frac{1}{pC}}{R + \frac{1}{pC}}} = \frac{\dfrac{R}{1 + pRC}}{pL + \dfrac{R}{1 + pRC}} =$$

$$= \frac{\dfrac{R}{1 + pRC}}{\dfrac{pL + p^2LCR + R}{1 + pRC}} = \frac{R}{pL + p^2LCR + R} =$$

$$= \frac{R}{LCR} \cdot \frac{1}{p^2 + \frac{1}{RC}p + \frac{1}{LC}} = \frac{1}{LC} \cdot \frac{1}{p^2 + \frac{1}{RC}p + \frac{1}{LC}} = \tag{7.59}$$

$$= \frac{\omega_0^2}{p^2 + 2\alpha p + \omega_0^2},$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}}, \ \alpha = \frac{1}{2RC}.$$

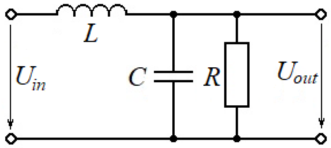



Fig. 7.28

Characteristic equation

$$p^2 + 2\alpha p + \omega_0^2 = 0.$$

Its roots

$$p_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}.$$

These roots can be both real and complex-conjugated.

In the general case, the filter of any order is formed by a cascade connection of separate filters of the first and second order with the elements of the decoupling between the links (fig. 7.29).

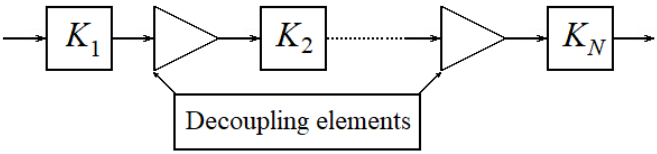



Fig. 7.29

In Fig. 7.29 $K_1$, $K_2$,..., $K_N$ – the coefficients of the transfer of links of the first and second order.

As a result, the transmission coefficient of the filter for rice. 6.29

$$K(p) = K_1(p) \cdot K_2(p) \cdot \ldots \cdot K_N(p).$$

**Example 6.7**.

It is necessary to realize the LPF with the maximum flat characteristic (Butterworth filter) of the third order with the cu toff frequency $\omega_c = 10^5 \frac{1}{\text{c}}$. The load of the filter is resistor $R = 0{,}5$ kOhm.

*Solution*.

Acc $c$ ording to the stages of synthesis: the requirements for the AFC of the LPF are stated in the task. Approximation of the AFC of the LPF on the condition of the Butterworth filter task. We record the transfer function of the third order filter. In general, it is recorded as follows

$$K(p_n) = \frac{1}{(p_n - p_{n1})(p_n - p_{n2}) \ldots (p_n - p_{n2n})}.$$

Define the coordinates of the poles of the transfer function. From Fig. 7.24,c for $n = 3$ it is evident that they are e poles 3.4.5, located in the left half-plane, that is,

$$p_{n3} = e^{j\frac{2\pi}{3}}, \; p_{n4} = -1, \; p_{n5} = e^{j\frac{4\pi}{3}}.$$

We renumber poles in order:

$p_{n1} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0{,}5 + j0{,}866;$

$p_{n2} = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0{,}5 - j0{,}866;$

$p_{n3} = -1.$

Previously, a replacement was made $j\omega$ on $p$ and $j\omega_n \to p_n$.

If $\omega_n = \frac{\omega}{\omega_c}$, then $\frac{j\omega}{\omega_c} = j\omega_n = \frac{p}{\omega_c} = p_n$.

From here

$p = p_n \cdot \omega_c$.

Turning now from the normalized variable $p_n$ to a real complex frequency, we get

$p_{1,2} = p_{n1} \cdot \omega_c = 10^5 \cdot (-0{,}5 \pm j0{,}866)$, $p_3 = p_{n3} \cdot \omega_c = -10^5$.

According to Fig. 7.29, the scheme of the third order filter can be constructed in the form of a cascade connection of the first order link with the pole $p_3$ and the second order links with the poles $p_{1,2}$ and the male solvers (Fig. 7.30).

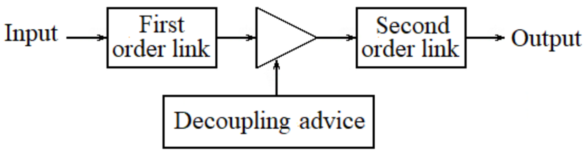



Fig.6.30

As a link of the first order choose a link according to the scheme in Fig. 7.27 than transfer coefficient by equation (7.58)

$$K(p) = \frac{1}{1 + p\tau},$$

where $\tau = RC$.

Frequency response is replaced $p$ by $j\omega$:

$$K(j\omega) = K(\omega)e^{j\varphi(\omega)} = \frac{1}{1 + j\omega\tau}.$$

Here

$$K(\omega) = \frac{1}{\sqrt{1 + (\omega\tau)^2}}.$$

At the cutoff frequency, the transmittance module in voltage decreases by $1/\sqrt{2}$ times. That is with $\omega = \omega_c$

$$\frac{1}{\sqrt{1+(\omega\tau)^2}} = \frac{1}{\sqrt{2}},$$

hence

$$1 + (\omega\tau)^2 = 2;\ \omega_c = \frac{1}{\tau};\ \tau = RC = \frac{1}{\omega_c}.$$

Choose arbitrarily $C = 10$ nF. Then

$$R = \frac{\omega_3}{C} = \frac{1}{10^5 \cdot 10 \cdot 10^{-9}} = 1000\ \Omega = 1\ \text{k}\Omega.$$

For a link of the second order, choose a link according to the scheme in Fig. 6.28. the role of the resistor $R$ performs in this case the load impedance $R_n$. Transfer coefficient according to the formula (7.59)

$$K(p) = \frac{\omega_0^2}{p^2 + 2\alpha p + \omega_0^2} = \frac{\omega_0^2}{(p-p_1)(p-p_2)}.$$

If

$$p_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = 10^5 \cdot (-0{,}5 \pm j0{,}866),$$

then

$$\alpha = \frac{1}{2R_{\text{н}}C} = 0{,}5 \cdot 10^5,$$

hence

$$C = \frac{1}{2R_{\text{н}} \cdot 0{,}5 \cdot 10^5} = \frac{1}{2 \cdot 0{,}5 \cdot 10^3 \cdot 0{,}5 \cdot 10^5} = 0{,}02\mu\text{F}.$$

Now, taking the resonant frequency of the series oscillatory circuit in the diagram of fig.7.28

$$\omega_0 = \frac{1}{\sqrt{LC}} = \omega_c,$$

get it

$$L = \frac{1}{\omega_3^2 C} = \frac{1}{(10^5)^2 \cdot 0{,}02 \cdot 10^{-6}} = 5 \cdot 10^{-3} = 5\ \text{mH}.$$

The schematic diagram of the synthesized LPF is shown in fig. 7.31.

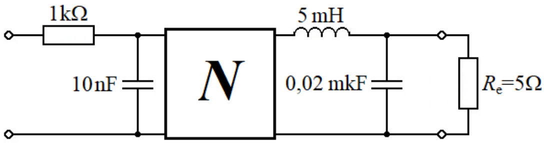


Fig. 6.31

As a soldering device, emitter or loop repeaters are usually used.

**Example 6.8.**

It is necessary to implement the Chebyshev second-order low-pass filter with a cut-off frequency $\omega_c = 10^5$ 1/s with a load $R_l = 1$ kΩ an irregularity coefficient ε = 1.

*Solution*.

According to the synthesis stages, the requirements for the amplitude-frequency characteristic (AFC) of the LPF under the conditions of the problem are carried out by a Chebyshev filter. We write the transfer function of the Chebyshev filter of the second order with the parameter ε = 1. In the general form

$$K(p_{\text{н}}) = \frac{1}{(p_{\text{н}} - p_{\text{н}1})(p_{\text{н}} - p_{\text{н}2})}.$$

Determine the coordinates of the poles of the transfer function. We calculate the auxiliary factor

$$a = \frac{1}{n}\ln\left(\frac{1}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2} + 1}\right) = \frac{1}{2}\ln\left(\frac{1}{1} + \sqrt{\frac{1}{1^2} + 1}\right) = 0{,}4407.$$

We calculate the poles of the second-order Butterworth LPFs. From Fig. 7.24,б it is evident that they have poles 2, 3 (in the left half plane), that is,

$$p_{n2} = e^{j\frac{3\pi}{4}}; p_{n3} = e^{j\frac{5\pi}{4}}.$$

Number the poles in order

$$p_{n1} = \cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4} = -0{,}707 + j0{,}707;$$
$$p_{n2} = \cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4} = -0{,}707 - j0{,}707.$$

Now define the abscissa (bonus of the abscissa of the Batteworth filter on the sh *a* and ordinate - on the ch *a*). From the tables we have:

$$\operatorname{sh} a = \operatorname{sh} 0{,}4407 = \frac{e^{0{,}4407} - e^{-0{,}4407}}{2} = 0{,}4551;$$
$$\operatorname{ch} a = \operatorname{ch} 0{,}4407 = \frac{e^{0{,}4407} + e^{-0{,}4407}}{2} = 1{,}0987.$$

Now the abscissas of the poles of the Chebyshev filter

$\operatorname{Re}[p'_{n1,2}] = \operatorname{Re}[p_{n1,2}] \cdot \operatorname{sh} a = (-0{,}707) \cdot 0{,}4551 = -0{,}322;$

Ordinates of the poles of Chebyshev's filter

$$\operatorname{Im}[p'_{n1,2}] = \operatorname{Im}[p_{n1,2}] \cdot \operatorname{ch} a = (\pm j0{,}707) \cdot 1{,}0987 = \pm j0{,}777.$$

As a result, the poles of the Chebyshev filter of the second order at $\varepsilon = 1$ are gaining shape

$$p'_{n1,2} = -0{,}322 \pm j0{,}777.$$

We move from the normalized variable $p_{\text{н}}$ to the real complex frequency

$$p'_{1,2} = p'_{n1,2} \cdot \omega_{\text{c}} = 10^5 \cdot (-0{,}322 \pm j0{,}777).$$

For the link of the second order we select the filter according to the scheme of fig. 7.28. The load is resistance $R_l$, The coefficient of transmission of the filter by the formula (7.59) is

$$K(p) = \frac{\omega_0^2}{p^2 + 2\alpha p + \omega_0^2} = \frac{\omega_0^2}{(p - p_1)(p - p_2)}.$$

If

$$p_{1,2} = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2} = 10^5 \cdot (-0{,}322 \pm j0{,}777),$$

then

$$\alpha = \frac{1}{2R_{\text{н}}C} = 0{,}322 \cdot 10^5,$$

where from

$$C = \frac{1}{2R_l \cdot 0{,}322 \cdot 10^5} = \frac{1}{2 \cdot 1000 \cdot 0{,}322 \cdot 10^5} =$$
$$= 15{,}53 \cdot 10^{-9} = 15{,}53\text{nF}.$$

Equate the imaginary parts of the poles

$$\sqrt{\omega_0^2 - \alpha^2} = 0{,}777 \cdot 10^5.$$

From here

$$\omega_0^2 - \alpha^2 = 0{,}6037 \cdot 10^{10};$$

$$\omega_0^2 = 0{,}6037 \cdot 10^{10} + \alpha^2 =$$

$$= 0{,}6037 \cdot 10^{10} + (0{,}322 \cdot 10^5)^2 = 0{,}7073 \cdot 10^{10}$$

If

$$\omega_0 = \frac{1}{\sqrt{LC}},$$

then

$$L = \frac{1}{\omega_0^2 C} = \frac{1}{0{,}7073 \cdot 10^{10} \cdot 15{,}53 \cdot 10^{-9}} = 9{,}1 \cdot 10^{-3} = 9{,}1\text{mH}.$$

The schematic diagram of the synthesized LPF is shown in fig. 7.32

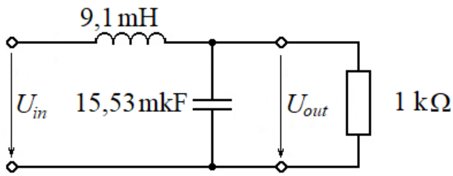



Fig. 6.32

## Methodical instructions

When studying the theory of filters it is necessary to use information about the characteristic parameters of the four-pole. To study the material of the section "Electric filters" is necessary on the example of the LPF. By the method of frequency transformation, expressions can be obtained for the remaining types of filters: the upper frequencies, the band and the barriers. You must understand the difference between "*k*" and"*m*" filters, as well as the benefits of "*m*" filters over "*k*" filters.

Synthesis of filters is carried out according to modern theory. Use approximation for Butterworth and Chebyshev. The approximation is given only by the amplitude-frequency characteristics. Filters are implemented on the basis of *RLC*-circuit of the first and second order.

Literatura: [1 - 4]

## Questions for self-examination

1. How do I distinguish filters by bandwidth location?

2. What are your relations for the coefficient of attenuation and the phase coefficient in the bandwidth and non-transmissibility for LPF?

3. How do filters like "*m*" get from filters like "*k*"?

4. What is the content of the operation of the normalization of resistance and frequency?

5. With what correlations for frequency it is possible to transform the LPF into high-frequency, SF, and ZF?

6. Output the stages of the LPF synthesis.

7. What are the approximations of the frequency characteristics of the LPF to you?

8. What is the content of the Butterworth filter approximation and the Chebyshev filter?

9. Describe the implementation of synthesized filters.

# 8. Circles with distributed parameters

## 8.1. Definition and equation of a long line

Circuits with distributed parameters (DPC) are idealized electric circuits, whose geometric dimensions exceed the wave length of transmitting electromagnetic oscillations. They differ from circles with lumped parameters so that the values of currents and voltage within the boundaries of the selected sections of the DPC do not remain unchanged, but change at the same time point from the intersection to the intersection.

Depending on the number of coordinate along which the current and voltage vary, one-dimensional, two-dimensional and three-dimensional DPC are distinguished. We will consider one-dimensional DPC, which are called long lines (LL), for example: communication lines, power lines.

A long line can be represented in the form of a set of continuously connected infinitesimal elements of length $dx$, each of which has its resistance $R_1 dx$, inductance $L_1 dx$, conductance $G_1 dx$ and capacitance $C_1 dx$ (fig.7.1). Resistance $R_1$, inductance $L_1$, conductance $G_1$, capacitance $C_1$ are the chassis parameters of LL per unit length. 1

If, on all sections LL $R_1 = \text{constant}$, $L_1 = \text{constant}$, $G_1 = \text{constant}$, $C_1 = \text{ constant}$ ie do not depend on coordinates, then LL is homogeneous or regular.

If $R_1 = 0$, $G_1 = 0$ that is, LL consists only of inductance $L_1$ and capacitance $C_1$, then it is called a loss free line (for example, power lines are modeled LL without losses).

If $L_1 = 0$, $G_1 = 0$, then the resulting line $R_1 C_1$ is used to simulate passive elements (film and diffusion resistors, capacitors, connecting conductors) of integrated microcircuits.

Parameters $R_1$, $L_1$, $G_1$, $C_1$are called primary parameters LL.

Consider the equation of LL. Let *x* - the distance from the beginning of the LL to the element *dx*, *i*, *u* – instantaneous values of current and voltage at the beginning of the element *dx*

The rate of change of current and voltage along the length of the line, obviously, can be written in the form $\frac{\partial i}{\partial x}$ and $\frac{\partial u}{\partial x}$. Then the current and voltage at the end of the element will be equal respectively $i + \frac{\partial i}{\partial x}dx$, $u + \frac{\partial u}{\partial x}dx$.

We will write for the node A (Fig. 8.1) the Kirchoff equation for currents, and for the loop, indicated by an arrow, the Kirchoff equation for voltage.

$$\begin{cases} i - \left(i + \frac{\partial i}{\partial x}dx\right) - \left(u + \frac{\partial u}{\partial x}dx\right)G_1 dx - C_1 dx\frac{\partial}{\partial x}\left(u + \frac{\partial u}{\partial x}dx\right) = 0; \\ -u + R_1 dxi + L_1 dx\frac{\partial i}{\partial t} + u + \frac{\partial u}{\partial x}dx = 0, \end{cases}$$

or

$$\begin{cases} i = \left(i + \frac{\partial i}{\partial x}dx\right) + \left(u + \frac{\partial u}{\partial x}dx\right)G_1 dx + C_1 dx\frac{\partial}{\partial t}\left(u + \frac{\partial u}{\partial x}dx\right); \\ u - \left(u + \frac{\partial u}{\partial x}dx\right) = R_1 dxi + L_1 dx\frac{\partial i}{\partial t}. \end{cases}$$

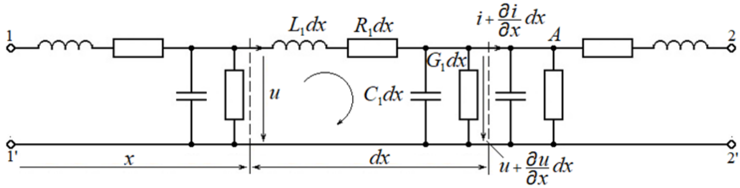



Fig. 8.1

Expanding the brackets, shortening on $dx$ and neglecting the terms with $(dx)^2$ as infinitesimal second order, we get

$$\begin{cases} u - u - \frac{\partial u}{\partial x}dx = R_1 dxi + L_1 dx\frac{\partial i}{\partial t}; \\ i = i + \frac{\partial i}{\partial x}dx + uG_1 dx + \frac{\partial u}{\partial x}dxG_1 dx + C_1 dx\frac{\partial u}{\partial t} + C_1 dx\frac{\partial}{\partial t}\frac{\partial u}{\partial x}dx. \end{cases}$$

Finally

$$\begin{cases} -\frac{\partial u}{\partial x} = R_1 i + L_1\frac{\partial i}{\partial t}; \\ -\frac{\partial i}{\partial x} = G_1 u + C_1\frac{\partial u}{\partial t}. \end{cases} \qquad (8.1)$$

Equations (8.1) are called differential equations of LL or telegraphic equations.

## 8.2 Long line with harmonious influence

We write the equation (8.1) in the operator form, moving from time *t* to the operator *p*:

$$\begin{cases} -\dfrac{dI(x,p)}{dx} = (G_1 + pC_1)\,U(x,p) - C_1 u(x,0); \\ -\dfrac{dU(x,p)}{dx} = (R_1 + pL_1)\,I(x,p) - L_1 i(x,0). \end{cases}$$

At zero initial conditions $u(x,0) = 0$, $i(x,0) \;=\; 0$

$$\begin{cases} -\dfrac{dI(x,p)}{dx} = Y_1(p)\,U(x,p); \\ -\dfrac{dU(x,p)}{dx} = Z_1(p)\,I(x,p), \end{cases} \tag{8.2}$$

where

$$Y_1(p) = G_1 + pC_1,\, Z_1(p) = R_1 + pL_1.$$

We differentiate the left and right sides of the second equation (8.2). We'll get it

$$\frac{dI(x,p)}{dx} = -\frac{1}{Z_1(p)}\frac{d^2U(x,p)}{dx^2}.$$

We substitute in the first equation (8.2)

$$\frac{d^2U(x,p)}{dx^2} = Z_1(p)Y_1(p)\,U(x,p) = \gamma^2(p)\,U(x,p),$$

or

$$\frac{d^2U(x,p)}{dx^2} - \gamma^2(p)\,U(x,p) = 0, \tag{8.3}$$

where $\gamma(p)$ - is the operator coefficient of propagation.

It is equal

$$\gamma(p) = \sqrt{Z_1(p)Y_1(p)} = \sqrt{(R_1 + pL_1)(G_1 + pC_1)}. \tag{8.4}$$

The general solution of equation (8.3), as an equation without the right-hand side, has the form

$$U(x,p) = A_1(p)e^{-\gamma(p)x} + A_2(p)e^{\gamma(p)x}, \tag{8.5}$$

where $A_1(p)$, $A_2(p)$ – the constants of integration (determined from the initial conditions at $x = 0$ (beginning of LL) and $x = 1$ (end of LL); $l$ – the length LL; $-\gamma(p)$, $\gamma(p)$ – the roots of the characteristic equation $p^2 + \gamma^2(p) = 0$, compiled for equation (8.3).

We differentiate the expression (8.5) by $x$ and substitute it in the second equation (8.2)

$$-A_1(p)e^{-\gamma(p)x}[-\gamma(p)] - A_2(p)e^{\gamma(p)x}\gamma(p) = Z_1(p)\, I(x,p).$$

From here

$$I(x,p) = \frac{A_1(p)e^{-\gamma(p)x}}{\frac{Z_1(p)}{\gamma(p)}} - \frac{A_2(p)e^{\gamma(p)x}}{\frac{Z_1(p)}{\gamma(p)}} = \frac{A_1(p)e^{-\gamma(p)x}}{Z_w(p)} - \frac{A_2(p)e^{\gamma(p)x}}{Z_w(p)}, \tag{8.6}$$

where $Z_w$ - the wave impedance.

It is equal

$$Z_w = \frac{Z_1(p)}{\gamma(p)} = \frac{Z_1(p)}{\sqrt{Z_1(p)Y_1(p)}} = \sqrt{\frac{Z_1(p)}{Y_1(p)}} = \sqrt{\frac{R_1 + pL_1}{G_1 + pC_1}}. \tag{8.7}$$

The system of equations (8.5) and (8.7) gives the value of operator voltages and currents of LL, depending on the coordinate $x$.

$$\begin{cases} \dot{U}(x) = \dot{A}_1 e^{-\dot{\gamma}x} + \dot{A}_2 e^{\dot{\gamma}x} = U_{fol}(x) + U_{ref}(x); \\ \dot{I}(x) = \frac{\dot{A}_1}{Z_w} e^{-\dot{\gamma}x} - \frac{\dot{A}_2}{Z_w} e^{\dot{\gamma}x} = I_{fol}(x) + I_{ref}(x). \end{cases} \tag{8.8}$$

For the analysis of processes in LL under harmonic influence, we rewrite the equations (8.5) and (8.6) in the complex form, replacing $p$ with $j\omega$:

$$\gamma = \sqrt{(R_1 + j\omega L_1)(G_1 + j\omega C_1)}, Z_w = \sqrt{\frac{R_1 + j\omega L_1}{G_1 + j\omega C_1}}. \tag{8.9}$$

where $\gamma_1$ - the coefficient of propagation, $Z_w$ – the wave impedance.

It is equal to:

$$\dot{\gamma} = \alpha + j\beta; \tag{8.10}$$

$$Z_w = Z_w e^{j\varphi}; \tag{8.11}$$

$$\dot{A}_1 = A_1 e^{j\psi_1}; \dot{A}_2 = A_2 e^{j\psi_2}. \tag{8.12}$$

We substitute the expressions (8.10) - (8.12) in equation (8.8)

$$\begin{cases} \dot{U}(x) = A_1 e^{j\psi_1} e^{-(\alpha+j\beta)x} + A_2 e^{j\psi_2} e^{(\alpha+j\beta)x}; \\ \dot{I}(x) = \frac{A_1 e^{j\psi_1} e^{-(\alpha+j\beta)x}}{Z_w e^{j\varphi}} - \frac{A_2 e^{j\psi_2} e^{(\alpha+j\beta)x}}{Z_w e^{j\varphi}}, \end{cases}$$

or through instantaneous values in the real form

$$\begin{cases} u(x,t) = \sqrt{2}A_1 e^{-\alpha x} cos(\omega t - \beta x + \varphi_1) + \\ \qquad +\sqrt{2}A_2 e^{\alpha x} cos(\omega t - \beta x + \varphi_2)\,; \\ i(x,t) = \dfrac{\sqrt{2}A_1 e^{-\alpha x}}{Z_w} cos(\omega t - \beta x + \psi_1 - \varphi) + \\ \qquad + \dfrac{\sqrt{2}A_2 e^{\alpha x}}{Z_w} cos(\omega t + \beta x + \psi_2 - \varphi)\,. \end{cases}$$

You can also write

$$\begin{cases} u(x,t) = u_{fol}(x,t) + u_{ref}(x,t); \\ i(x,t) = i_{fol}(x,t) - i_{ref}(x,t), \end{cases}$$

where

$$\begin{cases} u_{fol}(x,t) = \sqrt{2}A_1 e^{-\alpha x} \cos(\omega t - \beta x + \psi_1)\,; \\ u_{ref}(x,t) = \sqrt{2}A_2 e^{\alpha x} \cos(\omega t + \beta x - \psi_2); \end{cases} \tag{8.13}$$

$$\begin{cases} i_{fol}(x,t) = \frac{\sqrt{2}A_1 e^{-\alpha x}}{\bar{Z}_w} \cos(\omega t - \beta x + \psi_1 - \varphi); \\ i_{ref}(x,t) = -\frac{\sqrt{2}A_2 e^{\alpha x}}{\bar{Z}_w} \cos(\omega t + \beta x + \psi_2 - \varphi) = \\ \qquad = \frac{\sqrt{2}A_2 e^{\alpha x}}{\bar{Z}_w} \cos(\omega t + \beta x + \psi_2 - \varphi + \pi)\,, \end{cases} \tag{8.14}$$

where $u_{fol}$, $i_{fol}$, $u_{ref}$, $i_{ref}$ – falling and reflected waves of voltage and current.

Physical content of voltage  and current waives is as follows. Let's consider, the incident and reflected waves of the voltage. In the incident wave (fig. 8.2,a), with increasing of  *x*, one and the same voltage phase occurs at a greater value of *t*, that is, later. If you take it the starting point is the beginning of the line, then the maximum value of the wave over time shifted from the beginning of the line to its end: the wave of voltage as if moving from the beginning of the line. In the reflected wave (fig. 8.2, b), with increasing of  *x*, one and the same voltage phase occurs at a lower value of *t*, that is, before: the voltage wave moves from the end of the line to its beginning, returns. The amplitudes of the incident and reflected waves are reduced by exponentially in the direction of distribution (Fig. 8.2).

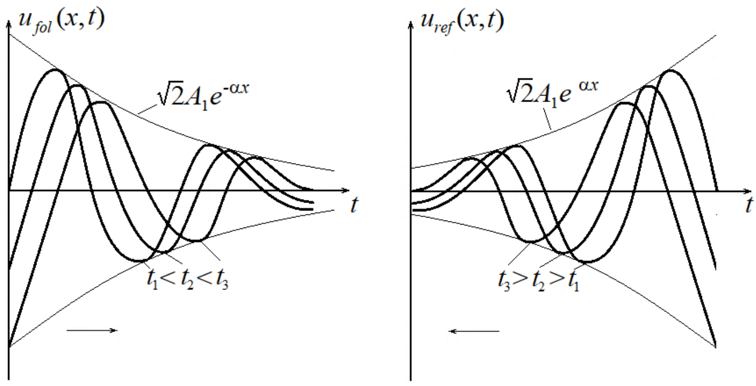



Fig.8.2

The value $\alpha = \mathrm{Re}[\gamma]$ is called the attenuation constant. Attenuation is caused by energy loss in $R_1$, $G_1$.

The value $\beta = \mathrm{Im}[\gamma]$ is called the phase coefficient and shows the phase change per unit length.

For a lossless line ($R_1 = 0$, $G_1 = 0$)

$$\alpha = 0; \gamma = j\omega\sqrt{L_1C_1}; \beta = \omega\sqrt{L_1C_1}. \qquad (8.15)$$

The distance between two points of the wave, the phases of which are different, is called the wave length. The wave length is found from the ratio

$$(\omega t - \beta x + \psi_1) - [\omega t - \beta(x + \lambda) + \psi_1] = 2\pi.$$

From here

$$\lambda = \frac{2\pi}{\beta}. \qquad (8.16)$$

For the lossless line, taking into account the expressions (8.15), we have

$$\lambda = \frac{2\pi}{\omega\sqrt{L_1C_1}} = \frac{1}{f\sqrt{L_1C_1}}. \qquad (8.17)$$

The velocity of moving along the line of the point with the same phase is called the phase velocity $\upsilon_{ph} = \frac{dx}{dt}$.

Speed is determined by the ratio

$$(\omega t - \beta x + \psi_1) = \mathrm{const}$$

or

$$\frac{d}{dt}(\omega t - \beta x + \psi_1) = 0,$$

that is

$$\omega - \beta \frac{dx}{dt} = 0; \ \ \upsilon_{ph.fol} = \frac{dx}{dt} = \frac{\omega}{\beta}.$$

In accordance

$$\upsilon_{ph.ref} = -\frac{\omega}{\beta}.$$

For the lossless line taking into account the expressions (8.15) we get

$$\upsilon_{ph} = \upsilon_{ph.fol} = |\upsilon_{ph.ref}| = \frac{1}{\sqrt{L_1 C_1}}. \qquad (8.18)$$

The phase velocity in the DL is close to the speed of light. Obviously, from the expressions (8.16), (8.17), (8.18)

$$\lambda = \frac{2\pi \upsilon_{ph}}{\omega} = \frac{\upsilon_{ph}}{f}.$$

Wavelength impedance $Z_w$ and propagation coefficient are called wave or secondary parameters of LL.

### 8.3. Reflection coefficient

The ratio of the voltage or currents of the reflected and incident waves in an arbitrary intersection of DL is called the reflection coefficient:

$$\dot{\rho}_u(x) = \frac{\dot{U}_{ref}(x)}{\dot{U}_{fol}(x)} = \frac{\dot{A}_2 e^{\dot{\gamma} x}}{\dot{A}_1 e^{-\dot{\gamma} x}} = \frac{\dot{A}_2}{\dot{A}_1} e^{2\dot{\gamma} x};$$

$$\dot{\rho}_i(x) = \frac{\dot{I}_{ref}(x)}{\dot{I}_{fol}(x)} = -\frac{\dot{A}_2 e^{\dot{\gamma} x} Z_w}{\dot{A}_1 e^{-\dot{\gamma} x} Z_w} = -\frac{\dot{A}_2}{\dot{A}_1} e^{2\dot{\gamma} x}. \qquad (8.19)$$

That is

$$\dot{\rho}_u(x) = -\dot{\rho}_i(x) = \dot{\rho}(x).$$

Let define integration constants $\dot{A}_1$, $\dot{A}_2$. Let’s put in equation (8.8) that $x = 0$. We’ll get it

$$\begin{cases} \dot{U}(0) = \dot{U}_1 = \dot{A}_1 + \dot{A}_2; \\ \dot{I}(0) = \dot{I}_1 = \dfrac{\dot{A}_1}{Z_w} - \dfrac{\dot{A}_2}{Z_w}, \end{cases} \tag{8.20}$$

where $\dot{U}_1$, $\dot{I}_1$ – voltage and current at the beginning of the line.

So,

$$\dot{A}_1 = \frac{\dot{U}_1 + \dot{I}_1 Z_w}{2}; \dot{A}_2 = \frac{\dot{U}_1 - \dot{I}_1 Z_w}{2}. \tag{8.21}$$

According to the expression (8.19)

$$\dot{\rho}(x) = \dot{\rho}_u(x) = \frac{\dot{U}_1 - \dot{I}_1 Z_w}{\dot{U}_1 + \dot{I}_1 Z_w} e^{2\dot{\gamma}x} = \dot{\rho}_1 e^{2\dot{\gamma}x}, \tag{8.22}$$

where $\dot{\rho}_1$ – reflection coefficient at the beginning of the line

$$\dot{\rho}_1 = \frac{\dot{U}_1 - \dot{I}_1 Z_w}{\dot{U}_1 + \dot{I}_1 Z_w} = \frac{\dfrac{\dot{U}_1}{\dot{I}_1} - Z_w}{\dfrac{\dot{U}_1}{\dot{I}_1} + Z_w} = \frac{Z_{11} - Z_w}{Z_{11} + Z_w}. \tag{8.23}$$

Consequently, the reflection coefficient at the beginning of the line is determined by the ratio between the input impedance of the line $Z_{11}$ and its wave impedance $Z_w$.

Integration constants $\dot{A}_1$, $\dot{A}_2$ can be determined by the voltage $\dot{U}_2$ and current $\dot{I}_2$ at the end of the line. We substitute in the expression (8.8) $x = l$:

$$\begin{cases} \dot{U}(l) = \dot{U}_2 = \dot{A}_1 e^{-\dot{\gamma}l} + \dot{A}_2 e^{\dot{\gamma}l}; \\ \dot{I}(l) = \dot{I}_2 = \dfrac{\dot{A}_1}{Z_w} e^{-\dot{\gamma}l} - \dfrac{\dot{A}_2}{Z_w} e^{\dot{\gamma}l}. \end{cases}$$

Now

$$\dot{A}_1 = \frac{\dot{U}_2 + \dot{I}_2 Z_w}{2} e^{\dot{\gamma}l}; \dot{A}_2 = \frac{\dot{U}_2 - \dot{I}_2 Z_w}{2} e^{-\dot{\gamma}l}.$$

Then, in accordance with the expression (8.19)

$$\dot{\rho}(x) = \dot{\rho}_u(x) = \frac{\dot{U}_2 - \dot{I}_2 Z_w}{\dot{U}_2 + \dot{I}_2 Z_w} e^{-2\dot{\gamma}(l-x)} = \dot{\rho}_2 e^{2\dot{\gamma}x'}, \tag{8.24}$$

where $\dot{\rho}_2$ – reflection coefficient at the end of the line; $x' = l - x$ – distance deduced from the end of the line

$$\dot{\rho}_2 = \frac{\dot{U}_2 - \dot{I}_2 Z_w}{\dot{U}_2 + \dot{I}_2 Z_w} = \frac{\frac{\dot{U}_2}{\dot{I}_2} - Z_w}{\frac{\dot{U}_2}{\dot{I}_2} + Z_w} = \frac{Z_l - Z_w}{Z_l + Z_w}. \qquad (8.25)$$

Thus, the reflection coefficient at the end of the line $\dot{\rho}_2$ is determined by its relation between the load resistance of the line $Z_l$ and its wave impedance $Z_w$.

From the formulas (8.22) and (8.24) we find the module of the coefficient of reflection

$$\mathrm{Mod}[\rho(x)] = \mathrm{Mod}\left[\rho_1 e^{2(\alpha+j\beta)x}\right] = \rho_1 e^{2\alpha x} =$$
$$\mathrm{Mod}\left[\rho_2 e^{-2(\alpha+j\beta)x'}\right] = \rho_2 e^{-2\alpha x'}. \qquad (8.26)$$

From expression (8.26) it is seen that the reflection coefficient module smoothly increases with height $x$ and reaches the highest value at the end of the line ($x = l$):

$$\rho_{max}(x) = \rho_1 e^{2\alpha l} = \rho_2.$$

For the lossless line (8.15), the reflection coefficient module retains the same value at all cross sections of the line

$$\rho = \rho_1 = \rho_2.$$

The voltage $\dot{U}(x)$ and current $\dot{I}(x)$ at any intersection of the line can be expressed through voltage $\dot{U}_1$, current $\dot{I}_1$ and reflection coefficient $\rho_1$ at the beginning of the line.

From formulas (8.8) and (8.21) we find

$$\dot{U}(x) = \frac{\dot{U}_1 + \dot{I}_1 Z_w}{2} e^{-\dot{\gamma}x} + \frac{\dot{U}_1 - \dot{I}_1 Z_w}{2} e^{\dot{\gamma}x}. \qquad (8.27)$$

From expression (8.23) we obtain

$$\dot{\rho}_1 \dot{U}_1 + \dot{\rho}_1 \dot{I}_1 Z_w = \dot{U}_1 - \dot{I}_1 Z_w; \quad \dot{I}_1 Z_w = \frac{1 - \dot{\rho}_1}{1 + \dot{\rho}_1} \dot{U}_1. \qquad (8.28)$$

We substitute expression (8.28) into formula (8.27). We have:

$$\dot{U}(x) = \frac{1}{2}\left[\dot{U}_1\left(e^{-\dot{\gamma}x} + e^{\dot{\gamma}x}\right) + \frac{1 - \dot{\rho}_1}{1 + \dot{\rho}_1} \dot{U}_1\left(e^{-\dot{\gamma}x} - e^{\dot{\gamma}x}\right)\right] =$$
$$= \frac{1}{2(1 + \dot{\rho}_1)}\left(2\dot{U}_1 e^{-\dot{\gamma}x} + 2\dot{U}_1 \dot{\rho}_1 e^{-\dot{\gamma}x}\right) = \qquad (8.29)$$
$$= \frac{e^{-\dot{\gamma}x} + \rho_1 e^{\dot{\gamma}x}}{1 + \rho_1} \dot{U}_1 = \frac{e^{-\dot{\gamma}x} + \rho_1 e^{\dot{\gamma}x}}{1 - \rho_1} \dot{I}_1 Z_w.$$

We obtain from formulas (8.8) and (8.21) :

$$\dot{I}(x) = \frac{\dot{U}_1 + \dot{I}_1 Z_w}{2Z_w} e^{-\dot{\gamma}x} - \frac{\dot{U}_1 - \dot{I}_1 Z_w}{2Z_w} e^{\dot{\gamma}x}. \qquad (8.30)$$

From the expressions (8.28) we obtain

$$\dot{U}_1 = \frac{1 + \dot{\rho}_1}{1 - \dot{\rho}_1} \dot{I}_1 Z_w. \qquad (8.31)$$

We substitute expression (8.31) into formula (8.30)

$$\dot{I}(x) =$$

$$= \frac{1}{2Z_w}\left[\left(\frac{1 + \dot{\rho}_1}{1 - \dot{\rho}_1} \dot{I}_1 Z_w + \dot{I}_1 Z_w\right) e^{-\dot{\gamma}x} - \left(\frac{1 + \dot{\rho}_1}{1 - \dot{\rho}_1} \dot{I}_1 Z_w - \dot{I}_1 Z_w\right) e^{\dot{\gamma}x}\right] = \qquad (8.32)$$

$$= \frac{e^{-\dot{\gamma}x} - \dot{\rho}_1 e^{\dot{\gamma}x}}{1 - \dot{\rho}_1} \dot{I}_1 = \frac{e^{-\dot{\gamma}x} - \dot{\rho}_1 e^{\dot{\gamma}x}}{(1 + \dot{\rho}_1) Z_w} \dot{U}_1.$$

From the expressions (8.29) and (8.32), by $x' = l - x$, $\dot{U}_1 = \dot{U}_2$, $\dot{I}_1 = \dot{I}_2$, having put we have formulas for currents and and voltages depending on, $x'$ ie, when counting the distance from the end of the line

$$\dot{U}(x) = \frac{e^{\dot{\gamma}x'} + \dot{\rho}_2 e^{-\dot{\gamma}x'}}{1 + \dot{\rho}_2} \dot{U}_2 = \frac{e^{\dot{\gamma}x'} - \dot{\rho}_2 e^{-\dot{\gamma}x'}}{1 - \dot{\rho}_2} \dot{I}_2 Z_w; \qquad (8.33)$$

$$\dot{I}(x) = \frac{e^{\dot{\gamma}x'} - \dot{\rho}_2 e^{-\dot{\gamma}x'}}{1 - \dot{\rho}_2} \dot{I}_2 = \frac{e^{\dot{\gamma}x'} - \dot{\rho}_2 e^{-\dot{\gamma}x'}}{(1 + \dot{\rho}_2) Z_w} \dot{U}_2. \qquad (8.34)$$

## 8.4. Wave modes

*Mode of running waves*. If the reflection coefficient $\rho(x)$ is zero, then the reflected wave will not be left, only the incident wave will remain. This is possible with formula (8.24) when $l = \infty$, the incident wave does not reach the end of the line, and therefore can not be reflected.

The mode of running waves also arises in a coordinated mode, if $Z_l = Z_w$.

Then in the formula (8.25) $\rho_2 = 0$ and in the formula (8.24) $\rho(x) = 0$. In the absence of a reflected wave ($U_{ref}(x) = 0$, $I_{ref}(x) = 0$) in the line remain only falling waves:

$$\begin{cases} \dot{U}(x) = \dot{U}_{fol}(x) = \dot{A}_1 e^{-\dot{\gamma}x}; \\ \dot{I}(x) = \dot{I}_{fol}(x) = \frac{\dot{A}_1}{Z_w} e^{-\dot{\gamma}x}. \end{cases} \quad (8.35)$$

Obviously, by the equation (8.20)

$$\dot{A}_1 = \dot{U}_1 = \dot{I}_1 Z_w \quad (8.36)$$

Then we have in the formula (8.35)

$$\begin{cases} \dot{U}(x) = \dot{U}_1 e^{-\dot{\gamma}x} = \dot{I}_1 Z_w e^{-\dot{\gamma}x}; \\ \dot{I}(x) = \dfrac{\dot{U}_1}{Z_w} e^{-\dot{\gamma}x} = \dot{I}_1 e^{-\dot{\gamma}x}. \end{cases}$$

That is, in running wave mode the amplitudes of the voltage and current exponentially decreases with height *x*. For the loss-free line ($\alpha = 0$) the amplitudes of the voltage and current remain unchanged in all sections of the LL.

In running waves the input impedance $Z_{11}$ is equal to the wave impedance according to equation (7.36)

$$Z_{11} = \frac{\dot{U}_1}{\dot{I}_1} = Z_w.$$

In running wave modes  energy is transmitted only in one direction – from the source  to the load.

*Standing wave mode*. If $Z_l \neq Z_w$ then only part of the energy transmitted by the incident wave is consumed by the load. The rest of the energy in the form of a reflected wave goes back to the source. If these energies are identical, a standing wave is established. Obviously that

$$\rho(\dot{x}) = 1. \quad (8.37)$$

From formula (8.24) we have

$$\rho(x) = \rho_2 e^{-2\alpha x'}.$$

Given the expression (8.37) we have

$$\rho_2 = 1, \alpha = 0, \quad (8.38)$$

that is, the mode of standing waves is set in a line without losses.

Given the value (8.38), we obtain from formula (8.25)

$$Z_l = 0; Z_l = \infty; Z_l = \mathrm{Im}[Z_l],$$

that is the mode of standing waves is established in a line without losses at short circuit or idling, as well as at a purely reactive load.

In the short circuit cuuutnt at the output ($Z_l = 0$) according to the formula (8.25) $\rho_2 = \rho_u = -1$, ie the voltage of the incident and reflected waves at the end of the LL are identical in amplitude, but are displaced by 180°. The instantaneous voltage output is zero.

For a current in accordance with expression (8.19)

$$\rho_2 = \rho_u = -1.$$

That is, the current of the incident and reflected waves at the end of the LL is the same in amplitude and coincides in phase. The instantaneous value of the current at the output is the maximum.

In the mode of standing waves, if $\alpha = 0$, then by equations (8.9) and (8.10)

$$Z_w = \sqrt{\frac{L_1}{C_1}} = R_w; \quad \dot{\gamma} = j\beta.$$

Then, with a short circuit at the output ($\rho_2 = -1$) from the expressions (8.33) and (8.34) we get

$$\dot{U}(x) = \frac{e^{j\beta x'} - e^{-j\beta x'}}{2}\dot{I}_2 R_w = j\dot{I}_2 R_w \sin(\beta x');$$

$$\dot{I}(x) = \frac{e^{j\beta x'} + e^{-j\beta x'}}{2}\dot{I}_2 = \dot{I}_2 \cos(\beta x') = \dot{I}_2 \operatorname{ch}(j\beta x').$$

Because of instantaneous values

$$\begin{cases} u(x,t) = \left[\sqrt{2}I_2 R_w \sin(\beta x')\right] \cos\left(\omega t + \frac{\pi}{2}\right); \\ i(x,t) = \left[\sqrt{2}I_2 \cos(\beta x')\right] \cos(\omega t), \end{cases}$$

that is, with a short circuit current at the output the amplitude of the voltage and current vary along the line according to the harmonic law:

$$\begin{cases} U_m(x) = \sqrt{2}I_2 R_w \sin(\beta x'); \\ I_m(x) = \sqrt{2}I_2 \cos(\beta x'). \end{cases} \tag{7.39}$$

The change in voltage $U_m(x)$ and current $I_m(x)$ is shown in fig. 8.3, a, b respectively. Points on the axis $x'$ (indicated by (•)), where the amplitudes of the voltage or current are zero, are called nodes. Points marked (×), where the amplitudes of the voltage and current are maximal, - the antinode. The location of nodes and antinodes on the axis $x'$ in time does not change. The wave is "standing" in the pleace. Therefore this mode is called standing wave mode.

From the above it is clear, that the nodes occur in those sections of the LL, where the voltages or currents of the incident and reflected waves are opposite to the phase and at the time of compilation give zero, and the antinodes - in those sections, where the voltages or currents coincide in phase and during compilation give maximum values.

In stationary waves, the active energy along the line is not transmitted, only the energy exchange between the electric and magnetic fields of LL occurs.

At idle of the output ($\rho_2 = 1$) of the equation (8.39) picks up

$$\begin{cases} U_m(x) = \sqrt{2}U_2 \cos(\beta x')\,; \\ I_m(x) = \sqrt{2}\dfrac{U_2}{R_w}\sin(\beta x'). \end{cases}$$

In this case, at the end of the line there will be the voltage antinode and the current node, the diagrams (Fig. 8.3) are shifted to the left or to the right for a quarter of the wave length.

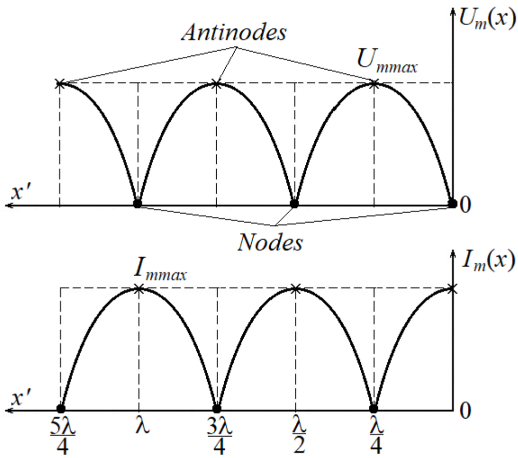



Fig. 8.3

If the load Impedance nce is purely reactive, then

$$Z_l = jx_l. \tag{8.40}$$

We substitute expression (8.40) in (8.25) for a loss less line

$$\dot{\rho}_2 = \frac{jx_l - R_w}{jx_l + R_w} = e^{j\varphi_{\rho 2}}, \tag{8.41}$$

where

$$\varphi_{\rho_2} = \begin{cases} \pi - 2\,\mathrm{arctg}\dfrac{x_l}{R_w} & x_l > 0; \\ -\pi - 2\,\mathrm{arctg}\dfrac{x_l}{R_w} & x_l < 0. \end{cases}$$

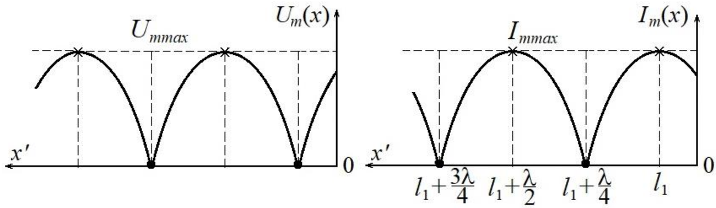
Then the voltage and current of the line from equations (8.33) and (8.34), using (8.41)

$$\begin{cases} \dot{U}(x) = \dot{U}_2\sqrt{1 + \left(\frac{x_l}{R_w}\right)^2}\cos(\beta x' - \varphi)\,; \\ \dot{I}(x) = -\dot{I}_2\sqrt{1 + \left(\frac{x_l}{R_w}\right)^2}\sin(\beta x' - \varphi)\,, \end{cases} \tag{8.42}$$

where

$$\varphi = \mathrm{arctg}\frac{x_l}{R_w}.$$

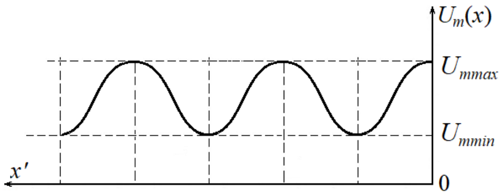
From expression (8.42) it can be seen that the amplitudes of voltage and current vary along the line according to the harmonic law. The points of the antinodes and the nodes of the voltage (Fig. 8.4, a) and current (Fig. 8.4, b) are shifted relative to the corresponding points for idle and short-circuit modes on $l_1 = \varphi\frac{\lambda}{2\pi}$. At the end of the line there are not nodes, and antinodes of voltage or current.



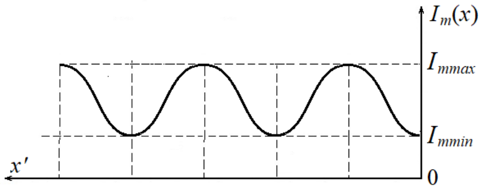

a b

Fig. 8.4

*Mixed wave mode*. The mode of mixed waves occupies an intermediate position between the modes of running and standing waves. Energy at the end of the line, wich transmitted by the wave, partially absorbed by the load and partly reflected. In this mode, the amplitudes of the voltage (Fig. 8.5, a) and current (Fig. 8.5, b) in the minima do not equal zero. The larger $U_{mmin}$, $I_{mmin}$ , the smaller the part of the energy is reflected from the load, and the stronger the mode of mixed waves will be different from the regime of standing waves.




a




b

Fig. 8.5

Valua

$$K_\delta = \frac{U_{mmin}}{U_{mmax}} = \frac{I_{mmin}}{I_{mmax}} \tag{8.43}$$

is called runway wave ratio (CRW)

$$0 \le K_\delta \le 1.$$

If

$$\begin{cases} U_{mmin} = U_{mfol} + U_{mref}, \\ U_{mmax} = U_{mfol} - U_{mref}. \end{cases}$$

Then, according to the expressions (8.19) and (8.43)

$$K_\delta = \frac{U_{mfol} - U_{mref}}{U_{mfol} + U_{mref}} = \frac{1 - \dfrac{U_{mref}}{U_{mfol}}}{1 + \dfrac{U_{mref}}{U_{mfol}}} = \frac{1-\rho(x)}{1+\rho(x)}.$$

Valua

$$K_C = \frac{1}{K_\delta} = \frac{1+\rho(x)}{1-\rho(x)}$$

is called standing wave coefficient(CSW) $\infty > K_C \geq 1$.

## 8.5. Transition processes in circles with distributed parameters

Let's consider the calculation of transient processes in circuit with distributed parameters on the example of LL. The calculation of transient processes is often reduced to the determination of the voltages and currents in different sections of the LL. Consider, for example, the distribution of voltages and currents in a homogeneous line without any loss at any external influence.

Let the voltage in the input line $u_1 = 0$ at $t < 0$, and when it changes at $t \geq 0$ according to the law $u_1(t)$, ie

$$u_1(t) = 1(t)u_1(t).$$

Let there also be a coordinated mode in the line, that is, the load impedance has an active resistance and is equal to the wave impedance

$$Z_l = R_w = \sqrt{\frac{L_1}{C_1}}.$$

We define the operator image of the voltage and current by expressions (8.5), (8.6). Integration constants $A_1(p)$ and $A_2(p)$ found at the beginning conditions ($x = 0$) and at the end ($x = 1$) of the line:

$$U(0,p) = U_1(p);\ \ U(l,p) = I(l,p)R_w \tag{8.44}$$

By $x = 0$ and $x = 1$ we obtain from the expressions (8.5) and (8.6)

$$U(0,p) = A_1(p) + A_2(p); \tag{8.45}$$

$$U(l,p) = A_1(p)e^{-\gamma(p)l} + A_2(p)e^{\gamma(p)l} \tag{8.46}$$

$$I(l,p) = \frac{A_1(p)}{R_w}e^{-\gamma(p)l} - \frac{A_2(p)}{R_w}e^{\gamma(p)l}. \tag{8.47}$$

We substitute (8.44) in the expressions (8.45) - (8.47)

$$A_1(p) = U_1(p);\quad A_2(p) = 0.$$

Then, according to equations (8.5) and (8.6)

$$\begin{cases} U(x,p) = U_1(p)e^{-\gamma(p)x} = U_1(p)e^{-\mathrm{p}\sqrt{L_1C_1}x}; \\ I(x,p) = \frac{U_1(p)}{R_w}e^{-\gamma(p)x} = I_1(p)e^{-\mathrm{p}\sqrt{L_1C_1}x}, \end{cases} \tag{8.48}$$

since for a loss less line ($R_1 = 0$, $G_1 = 0$) in accordance with expression (8.4)

$$\gamma(p) = p\sqrt{L_1C_1}.$$

In the system (8.48) $I_1(p) = I(0,p) = \frac{U_1(p)}{R_w}$ is the operator image of the current at the input LL.

Using the delay theorem, from the expression (8.48) we can conclude that the voltage and current in an arbitrary intersection of LL $u(x,t)$, $i(x,t)$ repeat the voltage and current at the beginning of the delay line $u_1$, $i_1$ for the period of time $t_x = \sqrt{L_1 C_1}x = \frac{x}{\upsilon_{ph}}$, for which the incident wave reaches the intersection $x$. The end of the line will reach this wave after a period of time

$$t_0 = \sqrt{L_1 C_1}I = \frac{l}{\upsilon_{ph}}.$$

Let at the moment $t = 0$ to input of the LL, opened at the end, connect the voltage $u_1(t) = E$. Then, by the equation (8.44)

$$U(0,p) = U_1(p) = \frac{E}{p};\;\; I(l,p) = 0. \qquad (8.49)$$

Substituting expressions (8.49) in the formula (8.45) - (8.47), we get

$$\begin{cases} \frac{E}{p} = A_1(p) + A_2(p); \\ \frac{A_1(p)}{R_w} e^{-\gamma(p)l} - \frac{A_2(p)}{R_w} e^{\gamma(p)l} = 0. \end{cases}$$

Where from

$$A_1(p) = \frac{1}{1 + e^{-2l\gamma(p)}}\frac{E}{p};\; A_2(p) = \frac{e^{-2l\gamma(p)}}{1 + e^{-2l\gamma(p)}}\frac{E}{p}. \quad (8.50)$$

From the formulas (8.5), (8.6) taking into account (8.50), we find:

$$U(x,p) = \frac{e^{-x\gamma(p)} + e^{-(2l-x)\gamma(p)}}{1 + e^{-2l\gamma(p)}}\frac{E}{p} = \frac{e^{-pt_x} + e^{-p(2t_0 - t_x)}}{1 + e^{-2pt_0}}\frac{E}{p}. \quad (8.51)$$

$$I(x,p) = \frac{e^{-x\gamma(p)} - e^{-(2l-x)\gamma(p)}}{1 + e^{-2l\gamma(p)}}\frac{E}{pR_w} = \frac{e^{-pt_x} - e^{-p(2t_0 - t_x)}}{1 + e^{-2pt_0}}\frac{I_0}{p}. \quad (8.52)$$

We portray the sum of an infinitely complicated geometric progression

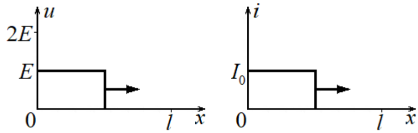
$$\frac{1}{1 + e^{-2pt_0}} = 1 - e^{-2pt_0} + e^{-4pt_0} - e^{-6pt_0} + e^{-8pt_0} - \cdots \quad (8.53)$$

Then from the expressions (8.51) and (8.52)

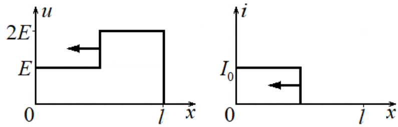
$$U(x,p) = \frac{E}{p}[e^{-pt_x} + e^{-p(2t_0 - t_x)} - e^{-p(2t_0 + t_x)} -$$
$$-e^{-p(4t_0 - t_x)} + e^{-p(4t_0 + t_x)} + e^{-p(6t_0 - t_x)} - \cdots]; \qquad (8.54)$$

$$I(x,p) = \frac{I_0}{p}[e^{-pt_x} - e^{-p(2t_0-t_x)} - e^{-p(2t_0+t_x)} +$$
$$+e^{-p(4t_0-t_x)} + e^{-p(4t_0+t_x)} - e^{-p(6t_0-t_x)} + \cdots]; \qquad (8.55)$$
$$u(x,t) = E[1(t-t_x) + 1(t-2t_0+t_x) - 1(t-2t_0-t_x) -$$
$$-1(t-4t_0+t_x) + 1(t-4t_0-t_x) + 1(t-6t_0+t_x) - \cdots]; \quad (8.56)$$
$$i(x,t) = I_0[1(t-t_x) - 1(t-2t_0+t_x) - 1(t-2t_0-t_x) +$$
$$+1(t-4t_0+t_x) + 1(t-4t_0-t_x) - 1(t-6t_0+t_x) - \cdots]. \quad (8.57)$$

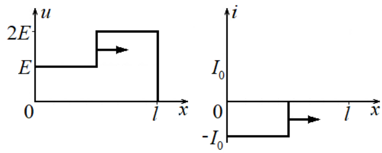
Expressions (8.56) and (8.57) show that the voltage and current at an arbitrary intersection of a line $x$ represent the sum of jumps, each of which appears at the moment of arrival at this point of the incident or reflected wave.




a




b

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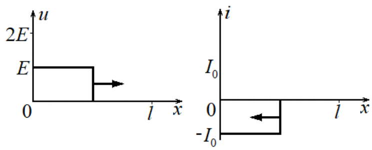
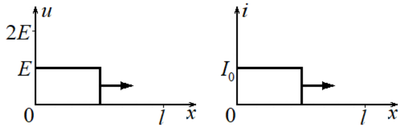



d




e

Fig. 8.6

The first jump (Fig. 8.6, a) arises at the moment $t_x$ of entry into this point of the incident wave (the first terms in the square brackets of the equations (8.56) and (8.57)), the second jump - through the time interval from the moment $2t_0 - t_x$ the transition occurs, when the point *x* arrives the wave, reflected from the load (the second terms in the square brackets of the equations (8.56), (8.57)). The reflection coefficient of the voltage at the end of the line is “$+1$”, and by the current “$-1$”, then the reflected wave arrives, is summed by voltage and subtracted by current (Fig. 8.6, b). The third jump occurs through the time interval from the moment $2t_0 + t_x$, when the wave arrives at the point $t = 0$, reflected from the source (third terms in square brackets) of the equations (8.56) and (8.57). The reflection coefficient of the voltage at the input of the line is equal to “$-1$”, and after the current “$+1$”. Thus, the reflected wave from the input is subtracted by voltage and current (Fig. 8.6, c).

Here it should be borne in mind that the internal resistance of the source is zero. The fourth jump arises due to the time $4t_0 - t_x$ when the wave arrives at the point *x*, again reflected from the load (the fourth plugs in the square brackets of the equations (8.56) and (8.57)).
If the coefficient of reflection at the end of the line, as indicated, is equal to 1 at the current “$-1$”, then the reflected wave is subtracted from the voltage and is added to the current (Fig. 8.6, d). This process is repeated (Fig. 8.6, e). Thus, at the end of the line, the current is always zero, and the voltage is $2E$ over a period of time $2t_0$, it is equal to zero for the same interval, that is, at the end of the open LL with no loss the voltage has the form of pulses (Fig.8.7). This property of the LL segment can be used in pulse shaper circuits.

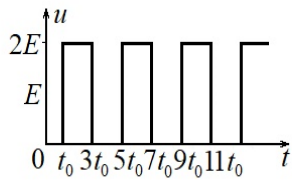


Fig. 8.7

Similarly transient processes can be considered in the short-circuited at the end of the line, which connects to the source of constant voltage.

In this case, the reflection coefficients of the voltage from the source of the input signal and at the end of the line are equal: “$-1$”, but by current “$+1$”. Therefore, at each reflection the voltage wave changes the sign, and the current wave does not change. As a result, the voltage at the end of the line always remains zero, and the current continuously increases. The successive stages of the transition process are shown in (Fig. 8. 8, a - c).

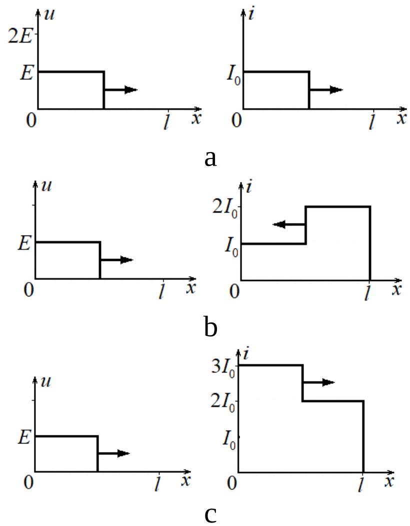



Fig. 8.8

In real lines with losses, the current in the line gradually approaches the set value. For a loss less line, the current increases with time along the stepped curve (Fig. 8, 9).

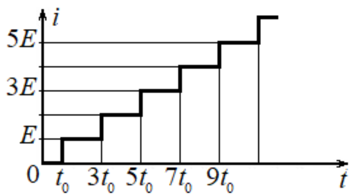


Fig. 8.9

## Methodical instructions

In the section " Circles with Distributed Parameters " you need to understand the basic difference between circuits with distributed parameters from circuits with lumped parameters: the values of currents and voltages in circuits with distributed parameters within the allocated sections of these circuits do not remain unchanged, but change for the same the time from the intersection to the intersection. This feature is represent  telegraph equations, in which currents and voltages depend not only on time but also on coordinates. Of course, there are no falling and reflected waves in real LLs, these are only a convenient abstraction for a clearer understanding of the processes in these circuits. This abstraction is particularly useful in analyzing transient processes in LL.

Literatura: [2 – 4], [9], [16]

### Questions for self-examination

1. What electrical circles are called circles with distributed parameters?
2. Record the telegraph equations of DL.
3. What is the incident and reflected wave of voltage and current?
4. How is the phase velocity determined in DL?
5. What is the reflection coefficient?
6. What are the known modes of waves and under what conditions they arise?
7. What is KTW and KSW?
8. Describe the transient process of current and voltage when switching on the DL to a constant voltage.